Topics in Applied Mathematics & Mathematical Physics

An Introduction to Monotonicity Methods for Nonlinear Kinetic Equations

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1. Introduction

Many nonlinear kinetic equations for complex systems appear as generalization of the classical Boltzmann equation (see, e.g. [4]). The last years have been marked by an increased interest in the mathematical properties of such models. This can be explained by various applications not only in physics, astrophysics and chemistry (e.g. studies of simple and complex/reacting fluids, granular media, coagulation-fragmentation, formation of planetary rings, galaxy collision) but also in modeling evolution processes in immunology, traffic flow, communication networks, etc.

In many situations, the above equations are phenomenological or microscopic models that describe the evolution of various *populations* (macroscopic systems) of many well individualized, *objects* (e.g. rarefied gas particles, cells networks signals etc.) interacting among themselves. The interactions are (localized) *microscopic* processes: a) any interaction has a very short duration, with respect to the time-scale of the macroscopic evolution; b) the number of partners of any interaction is very small, with respect to the total number of the components of the population. Depending on the model, an interaction may change the state, nature and/or the number of the participants in interaction. This may result in modifications of the values of the physical quantities characterizing the states of the interacting objects. However, such modifications must be consistent with certain *balance* laws (e.g. conservation /dissipation laws) imposed by the peculiarities of the microscopic processes.

The problem of the existence and uniqueness of solutions of the above models is not only of an academic interest. Indeed, good criteria for the existence of general solutions and a detailed study of the properties of the solutions can be particularly useful in obtaining effective convergent numerical schemes for the models.

The above models present some mathematical properties, similar to those of the classical Boltzmann equation, in particular similar monotonicity properties (with respect to the order). This made possible to extend nontrivially monotonicity methods, initially introduced for the classical Boltzmann equation, [2] (see also [28]) to study these models [18], [27], [9], [7]. Recently the ideas of [2] and [28]) have been reconsidered nontrivially within a more general, abstract framework, [11], [12], [13]. The present work is a survey of the recent progress in the domain, and includes five sections and an Appendix. This Introduction is the first Section. The next Section, is a brief presentation, at formal level, of some relevant examples of Boltzmann models for complex systems. In Section 3, we introduce a class of abstract evolution problems, as a generalization of the examples considered in Section 2. Then we develop the general existence theory based on monotonicity arguments. Section 4 is devoted to applications. Finally, Section 5 contains conclusions and open problems.

2. Boltzmann-like kinetic models

In this section we present several nonlinear models with nonlinear singularities, that exhibit similar isotonicity properties. In very general terms, these equations are essentially described by nonlinear evolution equations of the form

$$\frac{\mathrm{d}f}{\mathrm{d}t} = Af + Q(t,f), \quad t > 0, \tag{2.1}$$

formulated in the positive cone of some suitable ordered function space X, usually an ordered Banach space. The unknown $f = f(t)$ characterizes the state of the macroscopic system at time t. The two terms of the r.h.s. of Eq.(2.1), Af (possibly $A = 0$) and $Q(t,f)$ describe the free motion and the contribution of the interaction processes, respectively. From a mathematical point of view, A is the generator of a evolution linear group in X, while $Q(t,\cdot)$ is a nonlinear integral operator.

In many situations, we can write $Q(t,\cdot) = Q^+(t,\cdot) - Q^-(t,\cdot)$, where $Q^+(t,\cdot)$ and $Q^-(t,\cdot)$ are *positive* and *isotone* with respect to the order of X. Moreover, $Q^+(t,\cdot)$ and $Q^-(t,\cdot)$ satisfy certain relations -macroscopic balance laws-determined by the microscopic balance properties.

In this work we are interested in solving the initial value problem (i.v.p.) for Eq.(2.1), which can take various formulations, depending on the model.

2.1. Smoluchowski's coagulation equation

Smoluchowski's coagulation equation, [21, 25] (see also, e.g., [1], for a recent review), describes the irreversible evolution of particles that may coalesce into larger clusters. The continuous version of the Smoluchowski's equation reads

$$\frac{\partial}{\partial t} f = Q_c(f) = Q_c^+(f) - Q_c^-(f) \tag{2.2}$$

for the unknown $f(t,y) \geq 0$, the density of clusters of size $y \in \mathbb{R}_+ := [0,\infty)$ at time $t \geq 0$. Here

$$Q_c^+(g)(y) = \frac{1}{2}\int_0^y q(y-y_*,y_*)g(y-y_*)g(y_*)\mathrm{d}y_*, \tag{2.3}$$

$$Q_c^-(g)(y) = g(y)\int_0^\infty q(y,y_*)g(y_*)\mathrm{d}y_*, \tag{2.4}$$

with the (coagulation) kernel $q : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ a symmetric, measurable function.

We assume that there exist the constants $q_0, q_1 \geq 0$ and $0 \leq \alpha \leq \beta$, such that

$$q(y,y_*) \leq q_0 + q_1(y^\alpha y_*^\beta + y^\beta y_*^\alpha) \quad (y, y_* \geq 0), \tag{2.5}$$

where

$$\alpha + \beta \leq 1. \tag{2.6}$$

Condition (2.5) includes the case when either $q_0 = 0$ or $q_1 = 0$. Without loss of generality, we can assume that $q_1 > 0$ (indeed the situation when q is bounded by a constant can be considered as a particularization of (2.5) to the case where $q_1 > 0$ and $\alpha = \beta = 0$).

The following property of the Smoluchowski's model is essential for our analysis. Formally, if $g, \psi : \mathbb{R}_+ \mapsto \mathbb{R}$ are measurable, then

$$\int_0^\infty \psi(y)\left[Q_c^+(g)(y) - Q_c^-(g)(y)\right]\mathrm{d}y =$$

$$= \frac{1}{2}\int_0^\infty\int_0^\infty \widetilde{\psi}(y,y_*)q(y,y_*)g(y)g(y_*)\mathrm{d}y\mathrm{d}y_*, \tag{2.7}$$

(provided that the integrals exist), where

$$\widetilde{\psi}(y,y_*) := \psi(y+y_*) - \psi(y) - \psi(y_*). \tag{2.8}$$

Property (2.7) follows from the change of variables $(y,y_*) \to (y-y_*,y_*)$ in the first term of the l.h.s. of (2.7), and then applying Fubini's theorem.

In particular, if $\psi(y) = y$ in (2.7), then

$$\int_0^\infty Q_c(g)(y)y\mathrm{d}y = 0. \tag{2.9}$$

This gives formally the mass conservation for Eq. (2.2).

Similar considerations as before can be made for the discrete version of the Smoluchowski equation

$$\dot{c}_j = \frac{1}{2}\sum_{k=1}^{j-1} Q_{j-k,k}(c(t)) - \sum_{k=1}^{\infty} Q_{j,k}(c(t)), \quad c_j(0) = c_{j,0} \geq 0 \quad (j = 1, 2, ...), \tag{2.10}$$

where $Q_{j,k}(c) := q(k,j)c_k c_j$, is defined by the same symmetric coagulation kernel introduced before, subject to (2.5), (2.6), and the component $c_j(t) \geq 0$ of $c(t) := (c_j(t))$ is interpreted as the concentration of clusters of size j at time $t \geq 0$.

2.2. Povzner-like model with dissipative collisions

The model describes a rarefied mono-component fluid of particles of unit mass, evolving in the free space with dissipative (conservative) binary collisions, i.e., collisions resulting in the loss (conservation) of the kinetic energy of the encounters.

According to the model, [7], the post-collision velocities $\mathbf{v}'$, $\mathbf{w}'$ are related to the pre-collision velocities $\mathbf{v}$ and $\mathbf{w}$ by

$$\mathbf{v}^{'} = \mathbf{v} - (1-\beta(\mathbf{n}))\langle \mathbf{v}-\mathbf{w}, \mathbf{n}\rangle \mathbf{n}, \quad \mathbf{w}^{'} = \mathbf{w} + (1-\beta(\mathbf{n}))\langle \mathbf{v}-\mathbf{w}, \mathbf{n}\rangle \mathbf{n}, \tag{2.11}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean product in $\mathbb{R}^3$ and $\mathbf{n} \in \Omega$ - the unit sphere in $\mathbb{R}^3$. Here, $\beta : \Omega \mapsto [0, 1/2)$ is a given measurable function. The total momentum is conserved in collisions, $\mathbf{v}' + \mathbf{w}' = \mathbf{v} + \mathbf{w}$, but the kinetic energy is lost

$$\left|\mathbf{v}'\right|^2 + \left|\mathbf{w}'\right|^2 = |\mathbf{v}|^2 + |\mathbf{w}|^2 - 2\beta(\mathbf{n})(1-\beta(\mathbf{n}))\left|\langle \mathbf{v}-\mathbf{w}, \mathbf{n}\rangle\right|^2, \tag{2.12}$$

excepting the case $\beta = 0$, when the collisions become elastic.

For each fixed $\mathbf{n} \in \Omega$, the transformation $\mathbb{R}^3 \times \mathbb{R}^3 \ni (\mathbf{v},\ \mathbf{w}) \mapsto (\mathbf{v}', \mathbf{w}') \in \mathbb{R}^3 \times \mathbb{R}^3$ is invertible. The inversion formulae are

$$\hat{\mathbf{v}} = \mathbf{v} - \left(\frac{1-\beta(\mathbf{n})}{1-2\beta(\mathbf{n})}\right)\langle \mathbf{v}-\mathbf{w}, \mathbf{n}\rangle \mathbf{n}, \quad \hat{\mathbf{w}} = \mathbf{w} + \left(\frac{1-\beta(\mathbf{n})}{1-2\beta(\mathbf{n})}\right)\langle \mathbf{v}-\mathbf{w}, \mathbf{n}\rangle \mathbf{n}. \tag{2.13}$$

Formally the above model reads

$$\frac{\partial}{\partial t} f = -\mathbf{v} \cdot \nabla_{\mathbf{x}} f + Q_d^{+}(f) - Q_d^{-}(f) \tag{2.14}$$

where $f = f(t, \mathbf{x}, \mathbf{v})$ is the one-particle distribution function, depending on time $t \geq 0$, position $\mathbf{x} \in \mathbb{R}^3$, and velocity $\mathbf{v} \in \mathbb{R}^3$ of the so-called test particle,

Q_d^+ and Q_d^- are the so-called nonlinear gain and loss operators, respectively, and describe the influence of the collisions on the evolution of f. They are formally given by

$$Q_d^+(g)(\mathbf{x},\mathbf{v}) =$$
$$= \int_0^R \mathrm{d}r \int_{\Omega\times\mathbb{R}^3} \frac{|\langle \mathbf{n}, \mathbf{v}-\mathbf{w}\rangle|^\gamma}{(1-2\beta(\mathbf{n}))^{1+\gamma}} P(r,\mathbf{n}) g(\mathbf{x},\hat{\mathbf{v}}) g(\mathbf{x}+r\mathbf{n},\hat{\mathbf{w}}) \mathrm{d}\mathbf{n}\,\mathrm{d}\mathbf{w} \tag{2.15}$$

and

$$Q_d^-(g)(\mathbf{x},\mathbf{v}) = g(\mathbf{x},\mathbf{v}) \int_0^R \mathrm{d}r \int_{\Omega\times\mathbb{R}^3} |\langle \mathbf{n}, \mathbf{v}-\mathbf{w}\rangle|^\gamma P(r,\mathbf{n}) g(\mathbf{x}+r\mathbf{n},\mathbf{w}) \mathrm{d}\mathbf{n}\,\mathrm{d}\mathbf{w}, \tag{2.16}$$

respectively, where $P : \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}_+$ is a given measurable function with $P(r,\mathbf{n}) = P(r,-\mathbf{n})$ assumed to satisfy

$$P(r,\mathbf{n}) \leq c_0 r^2 \quad (r \geq 0,\ \mathbf{n} \in \Omega), \tag{2.17}$$

for some constants $c_0 > 0$, $0 \leq \gamma \leq 1$, and $R > 0$, specific to the collision processes.

The basic property of the model is the formal identity

$$\int_{\mathbb{R}^3} \psi(\mathbf{v}) \left[Q_d^+(g) - Q_d^-(g)\right] \mathrm{d}\mathbf{v} =$$
$$= \int_{\Omega\times\mathbb{R}^3\times\mathbb{R}^3} \widetilde{\psi}(\mathbf{v},\mathbf{w},\mathbf{v}',\mathbf{w}') \frac{|\langle \mathbf{n}, \mathbf{w}-\mathbf{v}\rangle|^\gamma}{2} P(r,\mathbf{n}) g(\mathbf{x},\mathbf{v}) g(\mathbf{x}+r\mathbf{n},\mathbf{w}) \mathrm{d}\mathbf{n}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}, \tag{2.18}$$

where $\psi : \mathbb{R}^3 \mapsto \mathbb{R}$ and $g : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}$ are measurable functions such that (2.18) is well defined, and

$$\widetilde{\psi}(\mathbf{v},\mathbf{w},\mathbf{v}',\mathbf{w}') := \psi(\mathbf{v}') + \psi(\mathbf{w}') - \psi(\mathbf{v}) - \psi(\mathbf{w}), \tag{2.19}$$

with $\mathbf{v}'$ and $\mathbf{w}'$ given by (2.11). We deduce easily (2.18), performing the change of variable $(v,w) \to (\hat{v},\hat{w})$ in the first term of the l.h.s (2.18).

If $\beta \equiv 0$, then (2.14) yields a version of the so-called generalized Boltzmann equation with binary elastic (conservative) collisions, analyzed in [3].

2.3. Povzner-like model with chemical reactions

We recall here a Povzner-like model with chemical reactions introduced in [8] for a reacting gas mixture of N species A_i and mass m_i, $1 \leq i \leq N$, without interaction with photon fields. We assume binary reactions

$$A_i + A_j \to A_k + A_l, \quad 1 \leq i,\ j,\ k,\ l \leq N, \tag{2.20}$$

where case $i = j = k = l$ corresponds to non-reactive (elastic) processes. According to the model of [8], for each species i, the gas particles have one internal energy state, say $E_i \geq 0$, $1 \leq i \leq N$. It is assumed that the reactions are consistent with the conservation of mass, momentum and total energy, i.e., $m_i + m_j = m_k + m_l$, and $m_i\mathbf{v} + m_j\mathbf{w} = m_k\mathbf{v}' + m_l\mathbf{w}'$, as well as

$$\frac{m_i\,|\mathbf{v}|^2}{2} + E_i + \frac{m_j\,|\mathbf{w}|^2}{2} + E_j = \frac{m_k\,|\mathbf{v}'|^2}{2} + E_k + \frac{m_l\,|\mathbf{w}'|^2}{2} + E_l, \qquad (2.21)$$

where $(\mathbf{v}, \mathbf{w})$ are the pre-reaction velocities of the particles (i, j) and $(\mathbf{v}', \mathbf{w}')$ are the post-reaction velocities of the particles (k, l)

The conservation relations give

$$\frac{m_k m_l\,|\mathbf{v}' - \mathbf{w}'|^2}{2(m_k + m_l)} = \frac{m_i m_j\,|\mathbf{v} - \mathbf{w}|^2}{2(m_i + m_j)} + E_i + E_j - E_k - E_l := t_{kl,ij}(\mathbf{v}, \mathbf{w}) \qquad (2.22)$$

and obviously, (2.20) occurs, provided that

$$t_{kl,ij}(\mathbf{v}, \mathbf{w}) \geq 0. \qquad (2.23)$$

It can be easily seen that $(\mathbf{v}', \mathbf{w}')$ can be represented in terms of the pre-reaction velocities $(\mathbf{v},\ \mathbf{w})$ and of the unit vector $\mathbf{n} = (\mathbf{v}' - \mathbf{w}')\,|\mathbf{v}' - \mathbf{w}'|^{-1}$ as

$$\mathbf{v}' = \frac{m_i\mathbf{v} + m_j\mathbf{w}}{m_i + m_j} + \frac{2^{1/2}(m_l)^{1/2}}{m_k^{1/2}(m_i + m_j)^{1/2}} t_{kl,ij}(\mathbf{v}, \mathbf{w})^{1/2}\mathbf{n} := \mathbf{v}_{kl,ij}(\mathbf{v}, \mathbf{w}, \mathbf{n}) \qquad (2.24)$$

and

$$\mathbf{w}' = \frac{m_i\mathbf{v} + m_j\mathbf{w}}{m_i + m_j} - \frac{2^{1/2}(m_k)^{1/2}}{m_l^{1/2}(m_i + m_j)^{1/2}} t_{kl,ij}(\mathbf{v}, \mathbf{w})^{1/2}\mathbf{n} := \mathbf{w}_{kl,ij}(\mathbf{v}, \mathbf{w}, \mathbf{n}) \qquad (2.25)$$

It is convenient to extend the definitions of $\mathbf{v}_{kl,ij}(\mathbf{v}, \mathbf{w}, \mathbf{n})$ and $\mathbf{w}_{kl,ij}(\mathbf{v}, \mathbf{w}, \mathbf{n})$ by setting

$$\mathbf{v}_{kl,ij}(\mathbf{v}, \mathbf{w}, \mathbf{n}) = \mathbf{w}_{kl,ij}(\mathbf{v}, \mathbf{w}, \mathbf{n}) = \frac{m_i\mathbf{v} + m_j\mathbf{w}}{m_i + m_j} \qquad (2.26)$$

whenever $t_{kl,ij}(\mathbf{v}, \mathbf{w}) < 0$. By virtue of the above formulae one has

$$\mathbf{v}_{kl,ij}(\mathbf{v}, \mathbf{w}, \mathbf{n}) = \mathbf{v}_{kl,ji}(\mathbf{w}, \mathbf{v}, \mathbf{n}) = \mathbf{w}_{lk,ij}(\mathbf{v}, \mathbf{w},\ -\mathbf{n}) \qquad (2.27)$$

and

$$\mathbf{w}_{kl,ij}(\mathbf{v}, \mathbf{w}, \mathbf{n}) = \mathbf{w}_{kl,ji}(\mathbf{w}, \mathbf{v}, \mathbf{n}) = \mathbf{v}_{lk,ij}(\mathbf{v}, \mathbf{w},\ -\mathbf{n}). \qquad (2.28)$$

Each species $1 \leq i \leq N$ is described by the one-particle distribution function $f_i = f_i(t, \mathbf{x}, \mathbf{v})$ depending on time $t \geq 0$, position $\mathbf{x}$ and velocity $\mathbf{v}$.

Assuming molecular chaos and (instant) point localized reactions, the kinetic model is derived following the original argument for the classical Boltzmann equation. The obtained model reads, [8],

$$\frac{\partial}{\partial t} f_i = -\mathbf{v} \cdot \nabla_{\mathbf{x}} f_i + Q_i^+(f) - Q_i^-(f), \quad 1 \leq i \leq N, \tag{2.29}$$

where $f = (f_1, ..., f_N)$ and, formally,

$$Q_i^+(g)(\mathbf{x}, \mathbf{v}) =$$

$$= \sum_{j,k,l=1}^{N} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} p_{kl,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n}) g_k(t, \mathbf{x}, \mathbf{v}_{kl,ij}) g_l(t, \mathbf{x} + \mathbf{y}, \mathbf{w}_{kl,ij}) \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{w}\mathrm{d}\mathbf{n}, \tag{2.30}$$

$$Q_i^-(g)(\mathbf{x}, \mathbf{v}) =$$

$$= \sum_{j,k,l=1}^{N} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \Omega} r_{kl,ij}(\mathbf{y},\ \mathbf{v}, \mathbf{w}, \mathbf{n}) g_i(t, \mathbf{x}, \mathbf{v}) g_j(t, \mathbf{x} + \mathbf{y}, \mathbf{w}) \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{w}\mathrm{d}\mathbf{n}. \tag{2.31}$$

Here, $g := (g_1, ...g_N)$ with $g_i : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$, $\Omega := \{\mathbf{n} \in \mathbb{R}^3 : |\mathbf{n}| = 1\}$, $g_k(\cdot, \cdot, \mathbf{v}_{kl,ij}) = g_k(\cdot, \cdot, \mathbf{v}_{kl,ij}(\mathbf{v}, \mathbf{w})$, $g_l(\cdot, \cdot, \mathbf{w}_{kl,ij}) = g_l(\cdot, \cdot, \mathbf{w}_{kl,ij}(\mathbf{v}, \mathbf{w}, \mathbf{n}))$. Moreover, $p_{kl,ij}, r_{kl,ij} : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \Omega \to [0, \infty)$, are given measurable maps with the property that if $(\mathbf{v}, \mathbf{w}) \notin \mathcal{D}_{ij,kl} := \{(\mathbf{v}, \mathbf{w}) \in \mathbb{R}^3 \times \mathbb{R}^3 : t_{ij,kl}(\mathbf{v}, \mathbf{w}) \geq 0\}$, then

$$p_{kl,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n}) = r_{kl,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n}) = 0. \tag{2.32}$$

One assumes that the following properties are satisfied a.e.:

$$p_{kl,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n}) = r_{kl,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n}) = 0 \quad (\mathbf{y} > R), \tag{2.33}$$

$$\begin{aligned} p_{kl,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n}) &= p_{kl,ij}(-\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n}), \\ r_{kl,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n}) &= r_{kl,ij}(-\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n}), \end{aligned} \tag{2.34}$$

$$p_{kl,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n}) = p_{kl,ji}(\mathbf{y}, \mathbf{w}, \mathbf{v}, \mathbf{n}) = p_{lk,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, -\mathbf{n}), \tag{2.35}$$

$$r_{kl,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n}) = r_{kl,ji}(\mathbf{y}, \mathbf{w}, \mathbf{v}, \mathbf{n}) = r_{lk,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, -\mathbf{n}). \tag{2.36}$$

Moreover,

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \Omega} \varphi(\mathbf{v}, \mathbf{w}) p_{kl,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n}) \psi(\mathbf{v}_{kl,ij}, \mathbf{w}_{kl,ij}) \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}\mathrm{d}\mathbf{n} =$$

$$= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\Omega} \varphi(\mathbf{v}_{ij,kl}, \mathbf{w}_{ij,kl}) r_{ij,kl}(\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n})\psi(\mathbf{v}, \mathbf{w}) \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}\mathrm{d}\mathbf{n} \tag{2.37}$$

for all $(\psi, \varphi) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, provided that whichever side of (2.37) is defined.

The kernels $p_{kl,ij}, r_{kl,ij} : \mathbb{R}^3 \times \mathbb{R}^3 \times \Omega \to [0, \infty)$ carry the information of the reaction processes. For a gas composed by one species of particles with elastic collisions, the above system of equations reduces to the so-called generalized Boltzmann equation.

Our main hypothesis is as follows:

Assumption 2.1 *There exist constants $c_q > 0$ and $0 \leq q \leq 1$ such that*

$$\int_\Omega r_{kl,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n})\mathrm{d}\mathbf{n} \leq c_q \left[1 + |\mathbf{v}|^2 + |\mathbf{w}|^2\right]^q. \tag{2.38}$$

Observe that since $r_{kl,ij}$ and $p_{kl,ij}$ are related by (2.37), then the above hypothesis is also an implicit condition on $p_{kl,ij}$.

Under Assumption (2.38), one can show that, at least, formally,

$$\sum_{i=1}^N \int_{\mathbb{R}^3\times\mathbb{R}^3} [Q_i^+(g)(\mathbf{x}, \mathbf{v}) - Q_i^-(g)(\mathbf{x}, \mathbf{v})] h_i(\mathbf{x}, \mathbf{v})\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} =$$

$$= \frac{1}{4} \sum_{i,j,k,l=1}^N \int_{\mathcal{D}} [p_{kl,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n}) g_k(\mathbf{x}, \mathbf{v}_{kl,ij}) g_l(\mathbf{x} + \mathbf{y}, \mathbf{w}_{kl,ij})$$

$$- r_{kl,ij}(\mathbf{y}, \mathbf{v}, \mathbf{w}, \mathbf{n}) g_i(\mathbf{x}, \mathbf{v}) g_j(\mathbf{x} + \mathbf{y}, \mathbf{w})]$$

$$\times [h_i(\mathbf{x}, \mathbf{v}) + h_j(\mathbf{x} + \mathbf{y}, \mathbf{w}) - h_k(\mathbf{x}, \mathbf{v}_{kl,ij}) - h_l(\mathbf{x} + \mathbf{y}, \mathbf{w}_{kl,ij})] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}\mathrm{d}\mathbf{n} \tag{2.39}$$

for all $g{=}(g_1, ...g_N)$ and $h{=}(h_1, ...h_N)$, with $g_i, h_i \geq 0$, for which the integrals are defined. Here, $\mathcal{D} := \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \Omega$. The last property follows by applying (2.27), (2.28), (2.32)–(2.37), as well as the invariance properties of the sums in (2.39), with respect to the change of variables $(\mathbf{x}, \mathbf{y}, \mathbf{n}) \to (\mathbf{x}', \mathbf{y}', \mathbf{n}') := (\mathbf{x} + \mathbf{y}, -\mathbf{y}, -\mathbf{n})$, and a suitable interchanges of summation indices.

At least, at formal level, property (2.39) implies the bulk conservation for mass, momentum, and total energy,

$$\sum_{i=1}^N \int_{\mathbb{R}^3\times\mathbb{R}^3} \Psi_i^{(j)}(\mathbf{x}, \mathbf{v}) f_i(t, \mathbf{x}, \mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} = \sum_{i=1}^N \int_{\mathbb{R}^3\times\mathbb{R}^3} \Psi_i^{(j)}(\mathbf{x}, \mathbf{v}) f_i(0, \mathbf{x}, \mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \tag{2.40}$$

($0 \le j \le 4$), where $f_i(t)$ are the components of the solution f of Eq. (2.29), and

$$\Psi_i^{(0)}(\mathbf{x},\mathbf{v}) := m_i, \quad \Psi_i^{(4)}(\mathbf{x},\mathbf{v}) := m_i\,|\mathbf{v}|^2/2 + E_i, \quad \Psi_i^{(j)}(\mathbf{x},\mathbf{v}) := m_i v_j \tag{2.41}$$

($j = 1, 2, 3$), with v_j are the components of $\mathbf{v}$.

2.4. A model with inelastic collisions and chemical reactions

In this example, we consider an abstract system of a Boltzmann-like phenomenological equations, [9, 10, 14], for a multi-component reacting gas of particles with internal states and discrete values of the internal energy. Thinking a real gas mixture of particles with internal structure as a mixture of several chemical species of mass points with unique internal state, one can assume that any gas particle of the model has only one internal state. Specifically, the model refers to a gas consisting of N chemical species. A particle of species $n = 1, 2, ..., N$ is characterized by mass $m_n > 0$ and internal energy E_n. Without loss of generality, one can assume that $E_n \ge 0$, $1 \le n \le N$. It is assumed that the chemical reactions are induced by inelastic (possibly) multi-body, instant collisions. A reaction is identified with a couple $(\alpha, \beta) \in \mathcal{M} \times \mathcal{M}$, where $\mathcal{M} := \{\gamma = (\gamma_n)_{1\le n\le N} \mid \gamma_n \in \{0, 1, \dots, K\}\}$ is a multi-index set. Here $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathcal{M}$ and $\beta = (\beta_1, \dots, \beta_N) \in \mathcal{M}$ designate the pre-collision and post-collision channels, respectively, with $0 \le \alpha_n, \beta_n \le K$ participants of species n; $1 \le n \le N$. Any couple of the form $(\gamma, \gamma) \in \mathcal{M} \times \mathcal{M}$ is identified with a multi-body elastic collision with γ_n collision partners of species n; $1 \le n \le N$. The number of particles in some channel $\gamma \in \mathcal{M}$ is $|\gamma| := \sum_{i=1}^{N} \gamma_i$. The family of chemical species participating in channel γ is denoted by $\mathcal{N}(\gamma) := \{i : \gamma_i > 0, 1 \le i \le N\}$.

Let M_γ, $\mathbf{V}_\gamma(\mathbf{w})$ and $W_\gamma(\mathbf{w})$ denote the total mass, velocity of the mass center and total energy, respectively, for the particles in channel γ, i.e.,

$$M_\gamma := \sum_{i=1}^{N} \gamma_i m_i, \tag{2.42}$$

$$\mathbf{V}_\gamma(\mathbf{w}) := \frac{1}{M_\gamma} \sum_{i\in\mathcal{N}(\gamma)} \sum_{j=1}^{\gamma_i} m_i \mathbf{w}_{i,j}, \tag{2.43}$$

$$W_\gamma(\mathbf{w}) := \sum_{i\in\mathcal{N}(\gamma)} \sum_{j=1}^{\gamma_i} (2^{-1} m_i \mathbf{w}_{i,j}^2 + E_i), \tag{2.44}$$

where $\mathbf{w} = ((\mathbf{w}_{k,i})_{i\in\{1,\dots,\alpha_k\}})_{k\in\mathcal{N}(\gamma)}$ represents the ensemble of velocities of the particles in channel γ. Then, the kinetic energy of the particles (with velocities $\mathbf{w}$) in channel γ, relative to the frame of the mass center, reads

$$W_{r,\gamma}(\mathbf{w}) = W_\gamma(\mathbf{w}) - \frac{M_\gamma \mathbf{V}_\gamma(\mathbf{w})^2}{2} - \sum_{i=1}^{N} \gamma_i E_i. \tag{2.45}$$

Obviously, $W_{r,\gamma}(\mathbf{w}) \geq 0$.

A gas reaction (α, β) may take place only if it is consistent with the conservation of mass, momentum and energy, i.e.,

$$M_\alpha = M_\beta, \quad \mathbf{V}_\alpha(\mathbf{w}) = \mathbf{V}_\beta(\mathbf{u}), \quad W_\alpha(\mathbf{w}) = W_\beta(\mathbf{u}). \tag{2.46}$$

We will assume here that elastic collisions are always present. Therefore, the set $\mathcal{C}_M := \{(\alpha,\beta) \in \mathcal{M}\times\mathcal{M} : M_\alpha = M_\beta\}$ is nonempty.

The Boltzmann-like system of equations for the above model is

$$\frac{\partial}{\partial t} f_i = Q_i^+(f) - Q_i^-(f). \tag{2.47}$$

Here the unknown $f_i : \mathbb{R}_+ \times \mathbb{R}^3 \mapsto \mathbb{R}_+$ is the one particle distribution functions $f_i = f_i(t, \mathbf{v})$ (t-time, $\mathbf{v}$-velocity) of the particles of species $1 \leq i \leq N$. In Eq. (2.47), $Q_i^+(f)$ and $Q_i^-(f)$, with $f := (f_1, \dots, f_N)$, are the so-called loss and gain (nonlinear) operators for the particles of species i, respectively. Formally,

$$Q_i^+(g)(\mathbf{v}) = \sum_{\alpha,\beta\in\mathcal{M}} \alpha_i \int\limits_{\mathbb{R}^{3|\alpha|-3}\times\Omega_\beta} \Big[p_{\beta,\alpha}(\mathbf{w},\mathbf{n})(g^\beta \circ \mathbf{u}_{\beta,\alpha})(\mathbf{w},\mathbf{n})\Big]_{\mathbf{w}_{i,\alpha_i}=\mathbf{v}} \mathrm{d}\tilde{\mathbf{w}}_i \mathrm{d}\mathbf{n}, \tag{2.48}$$

$$Q_i^-(g)(\mathbf{v}) = \sum_{\alpha,\beta\in\mathcal{M}} \alpha_i \int\limits_{\mathbb{R}^{3|\alpha|-3}\times\Omega_\beta} [r_{\beta,\alpha}(\mathbf{w},\mathbf{n}) g^\alpha(\mathbf{w})]_{\mathbf{w}_{i,\alpha_i}=\mathbf{v}} \mathrm{d}\tilde{\mathbf{w}}_i \mathrm{d}\mathbf{n}, \tag{2.49}$$

where

$$g^\gamma(\mathbf{w}) := \prod_{i\in\mathcal{N}(\gamma)} \prod_{j=1}^{\gamma_i} g_i(\mathbf{w}_{i,j}), \quad \gamma \in \mathcal{M}, \tag{2.50}$$

Ω_γ is the unit sphere in $\mathbb{R}^{3|\gamma|-3}$, with $\gamma \in \mathcal{M}$, and $\mathrm{d}\tilde{\mathbf{w}}_i$ is the Euclidean element of area on $\big\{\mathbf{w} \in \mathbb{R}^{3|\alpha|} \mid \mathbf{w}_{i,\alpha_i} = \mathbf{v}\big\}$. Here, the functions $\mathbf{u}_{\beta,\alpha} \in C(\mathbb{R}^{3|\alpha|} \times \Omega_\beta; \mathbb{R}^{3|\beta|})$, and the measurable functions $r_{\beta,\alpha}$, $p_{\beta,\alpha} : \mathbb{R}^{3|\alpha|} \times \Omega_\beta \mapsto \mathbb{R}_+$ are given.

The following conditions are assumed ([9, 11, 14]):

(B_1) $r_{\beta,\alpha} = p_{\beta,\alpha} = 0$ unless: $|\alpha| \geq 2$, $|\beta| \geq 2$, $(\alpha, \beta) \in \mathcal{C}_M$, and $\mathbf{w} \in D^+_{\beta,\alpha} := \left\{ \mathbf{w}' \in \mathbb{R}^{3|\alpha|} : W_{r,\alpha}(\mathbf{w}') + \sum_{i=1}^{N} (\alpha_i - \beta_i) E_i \geq 0 \right\}$.

(B_2) For each $i \in \mathcal{N}(\alpha)$ fixed, $p_{\beta,\alpha}(\mathbf{w}, \mathbf{n})$, $r_{\beta,\alpha}(\mathbf{w}, \mathbf{n})$, and $u_{\beta,\alpha}(\mathbf{w})$ are invariant with respect to the interchange of the components $\mathbf{w}_{i,1}, ..., \mathbf{w}_{i,\alpha_i}$ of $\mathbf{w}$.

(B_3) If $(\alpha, \beta) \in \mathcal{C}_M$, $\mathbf{w} \in D^+_{\beta,\alpha}$, then

$$(V_\beta \circ \mathbf{u}_{\beta,\alpha})(\mathbf{w}, \mathbf{n}) = V_\alpha(\mathbf{w}), \quad (W_\beta \circ \mathbf{u}_{\beta,\alpha})(\mathbf{w}, \mathbf{n}) = W_\alpha(\mathbf{w}), \tag{2.51}$$

for all $\mathbf{n} \in \Omega_\beta$, and

$$\int_{\mathbb{R}^{3|\alpha|} \times \Omega_\beta} p_{\beta,\alpha}(\mathbf{w}, \mathbf{n}) \varphi(\mathbf{w}, \mathbf{n}) (\psi \circ \mathbf{u}_{\beta,\alpha})(\mathbf{w}, \mathbf{n}) d\mathbf{w} d\mathbf{n} =$$

$$= \int_{\mathbb{R}^{3|\beta|} \times \Omega_\alpha} r_{\alpha,\beta}(\mathbf{w}, \mathbf{n}) (\varphi \circ \mathbf{u}_{\alpha,\beta})(\mathbf{w}, \mathbf{n}) \psi(\mathbf{w}, \mathbf{n}) d\mathbf{w} d\mathbf{n}, \tag{2.52}$$

for all $\varphi : \mathbb{R}^{3|\alpha|} \mapsto \mathbb{R}$ and $\psi : \mathbb{R}^{3|\beta|} \mapsto \mathbb{R}$, for which the integrals are well defined.

We suppose that the reactions are reversible, i.e., if $r_{\beta,\alpha} \neq 0$ for some (α, β), then also $r_{\alpha,\beta} \neq 0$.

From (3.9), it follows that $p_{\beta,\alpha}$ and $r_{\beta,\alpha}$ are related one to another. Indeed, a more explicit relationship between $p_{\beta,\alpha}$ and $r_{\beta,\alpha}$ can be derived, as it results from a general example constructed in [9, 14]. Note also here that if one assumes a mono-component gas of particles with binary elastic collisions (i.e., $N = 1$, $K = 2$, and $p_{\beta,\alpha} = r_{\beta,\alpha} = 0$ unless $\alpha = \beta = (1, 1)$), then Eq. (2.47) reduces to the space homogeneous classical Boltzmann equation

$$\frac{\partial}{\partial t} f = Q^+(f) - Q^-(f), \tag{2.53}$$

where

$$Q^+(f)(\mathbf{v}) = \int_{\mathbb{R}^3 \times \Omega} q(\mathbf{v}, \mathbf{w}, \mathbf{n}) f(\mathbf{v}') f(\mathbf{w}') d\mathbf{w} d\mathbf{n}, \tag{2.54}$$

$$Q^-(f)(\mathbf{v}) = \int_{\mathbb{R}^3 \times \Omega} q(\mathbf{v}, \mathbf{w}, \mathbf{n}) f(\mathbf{v}) f(\mathbf{w}) d\mathbf{w} d\mathbf{n}. \tag{2.55}$$

The notations are $f = f(t, \mathbf{v})$ – the one-particle distribution function, $\mathbf{v}' = \mathbf{v} - \langle \mathbf{v} - \mathbf{w}, \mathbf{n} \rangle \mathbf{n}$, $\mathbf{w}^{'} = \mathbf{w} + \langle \mathbf{v} - \mathbf{w}, \mathbf{n} \rangle \mathbf{n}$, and $\mathbf{n} \in \Omega$ – the unit sphere in $\mathbb{R}^3$. Here, the Boltzmann collision law q is a positive measurable function (depending, in our case, on $\mathbf{v}$ and $\mathbf{w}$ through the variable $\mathbf{v} - \mathbf{w}$).

The last condition of the model concerns the behavior of $r_{\beta,\alpha}$ (see [9]):

Assumption 2.2 *There are some constants $0 \le q \le 1$ and $c_q > 0$ such that*

$$\nu_{\beta,\alpha}(\mathbf{w}) := \int_{\Omega_\beta} r_{\beta,\alpha}(\mathbf{w}, \mathbf{n}) \mathrm{d}\mathbf{n} \le c_q (1 + W_\alpha(\mathbf{w}))^q \quad (\mathbf{w} \in \mathbb{R}^{|\alpha|},\ a.e.), \tag{2.56}$$

for all $\alpha, \beta \in \mathcal{M}$.

Obviously, $\nu_{\beta,\alpha}(\mathbf{w}) = 0$, unless $(\alpha, \beta) \in \mathcal{C}_M$.

A consequence of (B_1), (B_2) and (2.56) is the key equality

$$\sum_{i=1}^{N} \int_{\mathbb{R}^3} \Psi_i^{(j)}(\mathbf{v}) \left[Q_i^+(g)(\mathbf{v}) - Q_i^-(g)(\mathbf{v}) \right] \mathrm{d}\mathbf{v} = 0 \quad (0 \le j \le 4), \tag{2.57}$$

for all $g = (g_1, ..., g_N)$ with $(1 + |\mathbf{v}|^2)^{1+q} g_i \in L^1(\mathbb{R}^3; \mathrm{d}\mathbf{v})$, $i = 1, 2, ..., N$. Here,

$$\Psi_i^{(0)}(\mathbf{v}) := m_i, \quad \Psi_i^{(4)}(\mathbf{v}) := \frac{1}{2} m_i |\mathbf{v}|^2 + E_i, \quad \Psi_i^{(j)}(\mathbf{v}) := m_i v_j \quad (1 \le i \le N), \tag{2.58}$$

where v_j is the j-component, $j = 1, 2, 3$, of $\mathbf{v}$. Equality (2.57) implies, at lest formally, the bulk conservation of mass, momentum and total energy.

2.5. A nonlinear von Neumann-Boltzmann equation

Besides classical models, we can also consider "quantum" kinetic models with monotonicity properties similar to classical ones.

Let $X = \mathcal{T}(\mathcal{H})$ be the space of trace class selfadjoint operators in some separable Hilbert space $\mathcal{H}$. On X, we consider the order $F \le G$ iff $(f, Ff) \le (f, Gf)$, $\forall f \in \mathcal{D}(F) \cap \mathcal{D}(G)$. Let $\|F\| := Tr(|F|)$ be the norm on X.

For some orthogonal base $\{e_0, e_1, ...\} \subset \mathcal{H}$, define the selfadjoint operator

$$H = \sum_{i \ge 0} \mu_i (e_i, \cdot) e_i, \tag{2.59}$$

where $\{\mu_n\}_n \subset \mathbb{R}$. Let $\{U^t\}_{t\in\mathbb{R}}$ denote the continuous group of positive isometries on X, given by $U^t(F) := \exp(-\mathrm{i}Ht)F\exp(\mathrm{i}Ht)$, $\mathrm{i} = \sqrt{-1}$. Consider a second sequence, $0 \leq \lambda_0 < \lambda_1 < \lambda_2 \leq \ ... \ \lambda_{n-1} \leq \lambda_n \ ... \ \nearrow \infty$, as $n \to \infty$. Let $\left\{V^t\right\}_{t\geq 0}$ be the C_0 semigroup on X, defined by

$$(e_i, V^t(F)e_j) := (V^t(F))_{i,j} = \exp[-(1+\lambda_i\delta_{i,j})t]F_{i,j} \tag{2.60}$$

where $F_{i,j} := (e_i, Fe_j)$, and let the infinitesimal generator of $\left\{V^t\right\}_{t\geq 0}$ be denoted by $(-\Lambda)$. Then

$$(\Lambda)_{i,j}(F) := (1+\lambda_i\delta_{i,j})F_{i,j}, \tag{2.61}$$

hence $\Lambda \geq \mathbb{I}$. Clearly, U^t leaves $\mathcal{D}(\Lambda)\cap X_+$ invariant and $U^t\Lambda = \Lambda U^t$ on $\mathcal{D}(\Lambda)\cap X_+$.

Now we can consider the following example of nonlinear von Neumann-Boltzmann equation X (see also [12]):

$$\frac{\mathrm{d}F}{\mathrm{d}t} + \mathrm{i}[H,F] = Q^+(F) - Q^-(F) \tag{2.62}$$

with $Q^\pm : \mathcal{D}(\Lambda) \subset X \to X$ given by

$$Q^-(F) := F_{0,0}\mathrm{Tr}(\Lambda F)(\sum_{i=0}^{2} P_i), \tag{2.63}$$

and

$$Q^+(F) := Q^-(F) + L(F), \tag{2.64}$$

where $P_i := (e_i, \cdot)e_i$ and

$$L(F) := F_{0,0}\mathrm{Tr}(\Lambda F)(\sum_{i=0}^{2} \varepsilon_i P_i). \tag{2.65}$$

Here, $\varepsilon_0 = \varepsilon\,(\lambda_1-\lambda_0)^{-1}\,(\lambda_2-\lambda_0)^{-1}$, $\varepsilon_1 = -\varepsilon(\lambda_1-\lambda_0)^{-1}\,(\lambda_2-\lambda_1)^{-1}$, $\varepsilon_2 = \varepsilon(\lambda_2-\lambda_0)^{-1}\,(\lambda_2-\lambda_1)^{-1}$ and $0 < \varepsilon < (\lambda_0-\lambda_1)\,(\lambda_0-\lambda_2)$. Thus $Q^\pm$ are positive operators, and a simple computation gives

$$\mathrm{Tr}Q^+(F) = \mathrm{Tr}Q^-(F) \tag{2.66}$$

for $0 \leq F \in \mathcal{D}(\Lambda)$, and

$$\mathrm{Tr}(\Lambda Q^+)(F) = \mathrm{Tr}(\Lambda Q^-)(F) \tag{2.67}$$

for $0 \leq F \in \mathcal{D}(\Lambda^2)$, so that both $\mathrm{Tr}F(t)$ and $\mathrm{Tr}(\Lambda F)(t)$ remain constant with time.

3. General theory

3.1. A monotonicity result for the classical Boltzmann equation

Before proceeding to a more general analysis, we start with a relevant example - the Arkeryd's monotonicity result for the Boltzmann equation ([2]).

Specifically, in [2], the main interest is to solve the Cauchy problem for the space homogeneous Boltzmann equation (2.47) in the positive cone L^1_+ of $L^1 = L^1(\mathbb{R}^3, \mathrm{d}\mathbf{v})$, namely

$$\frac{\mathrm{d}}{\mathrm{d}t} f = Q(f) \equiv Q^+(f) - Q^-(f),\ f(0) = f_0 \geq 0\ (t \geq 0) \tag{3.1}$$

with $Q^\pm$ defined by (2.54) and (2.55), respectively.

The basic hypothesis is that the collision kernel q satisfies

$$q(\mathbf{v}, \mathbf{w}, \mathbf{n}) \leq C_q(1 + |\mathbf{v}|^\lambda + |\mathbf{w}|^\lambda) \quad (0 \leq \lambda \leq 2), \tag{3.2}$$

for some constant $C_q > 0$. The initial data f_0 is supposed to satisfy (at least) the condition of finite mass and energy, i.e. $\|f_0\|_2 < \infty$, where

$$\|g\|_l := \int (1 + |\mathbf{v}|^2)^{\frac{l}{2}} |g(\mathbf{v})| \, \mathrm{d}\mathbf{v}. \tag{3.3}$$

Unfortunately, under condition (3.2), the operators $Q^\pm$ are too singular to allow for applying general methods to the above problem. The idea of [2] is to approximate $Q^\pm$ by collision-like operators $Q^\pm_m$ with bounded (hence simpler) kernels $q_m(\mathbf{v}, \mathbf{w}) := \min\{q(\mathbf{v}, \mathbf{w}), m\}$, $m = 1, 2, \ldots$.

Thus one starts by solving the simple model

$$\frac{\mathrm{d}}{\mathrm{d}t} f = Q_m(f) \equiv Q^+_m(f) - Q^-_m(f),\ f(0) = f_0\ (t \geq 0). \tag{3.4}$$

Note that, since (3.4) is a Boltzmann-type equation, then for "many" $g \in L^1$,

$$\int \varphi_i(\mathbf{v}) Q_m(g) \mathrm{d}\mathbf{v} = 0, \tag{3.5}$$

where $\varphi_0(\mathbf{v}) = 1$, $\varphi_i(\mathbf{v}) = \mathbf{v}_i$, $i = 1, 2, 3$, $\varphi_4(\mathbf{v}) = |\mathbf{v}|^2$. An immediate consequence is that for any solution $f = f(t, \mathbf{v})$ of (3.4),

$$\|f(t)\|_0 = \|f_0\|_0\ (t \geq 0). \tag{3.6}$$

Moreover, if also $\|f(t)\|_2 < \infty$, then

$$\|f(t)\|_2 = \|f_0\|_2 \,. \tag{3.7}$$

Writing the solution of (3.4) as f_m, one could hope that if $m \to \infty$, then f_m converges somehow to a solution of the original problem (3.1). Another key point in the analysis is to use the above equalities as a priori estimates in order to replace (3.4) with other (somehow equivalent) equations, more suitable for monotone iteration with respect to the natural order of L^1.

Thus, one can first prove the following result ([2]).

Proposition **3.1** *There exists a unique non-negative solution* $f_m(t, \mathbf{v}) \in L^1$ *of* (3.4) *for every* $0 \leq f_0 \in L^1$.

Proof. By (3.6), the positive solutions (in L^1) of (3.4) are exactly the positive solutions of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f + C \|f_0\|_0 \, f = Q_m(f) + C \|f(t)\|_0 \, f, \; f(0) = f_0 \; (t \geq 0), \tag{3.8}$$

which satisfy equality (3.6). Here $C > 0$ is some constant. Let $v(t) := \exp(-C \|f_0\|_0 \, t)$. Since the operators $Q_m^{\pm}$ are locally Lipschitz in L^1, (3.8) has a unique local solution $f_m(t)$, which is also a unique local solution to the mild equation

$$f(t) = v(t) f_0 + \int_0^t v(t-s)[Q_m(f)(s) + C \|f(s)\|_0 \, f(s)] \mathrm{d}s. \tag{3.9}$$

Define the sequence $\{f_m^n\}_n$ by

$$f_m^1 = 0, \quad f_m^n = v(t) f_0 + \int_0^t v(t-s)[Q_m(f_m^n)(s) + C \|f_m^n(s)\|_0 \, f_m^n(s)] \mathrm{d}s. \tag{3.10}$$

If C is sufficiently large, then the operator $X \ni g \to Q_m(g) + C \|g\|_0 \, g \in X$ is positive. Then the sequence $\{f_m^n(t)\}_n$ is positive and increasing in L^1. A simple induction, making use of (3.5), gives $\|f_m^n(t)\|_0 \leq \|f_0\|_0$. Then by the monotone completeness of L^1 (Levi's theorem) $\{f_m^n(t)\}_n$ is convergent, its limit $g^m(t)$ satisfies (3.9), and $\|g_m(t)\|_0 \leq \|f_0\|_0$. But by virtue of the uniqueness of the aforementioned local solution $f_m(t)$ (of both (3.8) and (3.9)), clearly $g_m(t) = f_m(t) \geq 0$ for t small enough. Moreover, $g^m(t)$ extends $f_m(t)$, as the unique solution of (3.8), for all $t \geq 0$. It remains to show that this solution satisfies (3.6). To this end, one integrates (3.8), with f_m as solution, and rearrange conveniently the resulting expression as

$$f_m + \int_0^t [Q_m^-(f_m)(s) + C \|f_0\|_0 f_m(s)]\mathrm{d}s =$$

$$= f_0 + \int_0^t [Q_m^+(f_m)(s) + C \|f_m(s)\|_0 f_m(s)]\mathrm{d}s. \tag{3.11}$$

As $f_m(t)$, $Q_m^{\pm}(f_m)(t) \geq 0$, invoking the additivity of the L^1 norm, and the property $\|f_m(t)\|_0 \leq \|f_0\|_0$, one finally obtains

$$0 \leq \|f_0\|_0 - \|f_m(t)\|_0 \leq C \|f_0\|_0 \int_0^t (\|f_0\|_0 - \|f_m(s)\|_0)\mathrm{d}s. \tag{3.12}$$

Thus by Gronwall's inequality,

$$\|f_m(t)\|_0 = \|f_0\|_0 , \quad (t \geq 0) \tag{3.13}$$

so the proof is concluded. □

An induction involving (3.10), and making use of (3.5) also shows ([2]) that if f_m is as in Prop. 3.1, and $(1 + |\mathbf{v}|^2)f_0 \in L^1$, then $(1 + |\mathbf{v}|^2)f_m \in L^1$, and

$$\|f_m(t)\|_2 = \|f_0\|_2 \ (t \geq 0). \tag{3.14}$$

Another important property is the following estimation, uniform with respect to m (see [2]): for any $t_* > 0$,

$$\|f_m(t)\|_l \ \leq K \|f_0\|_l \quad (0 \leq t \leq t_*), \quad l \geq 4, \tag{3.15}$$

for some number $0 < K = K(t_*, \|f_0\|_2 , C_q, l)$. The proof (see the slightly more general Prop. 1.3 of [2]) is inductive, and applies (3.10) and the basic inequality

$$\int_{\mathbb{R}^3} (1 + |\mathbf{v}|^2)^{\frac{l}{2}} Q_m(f_m)\mathrm{d}\mathbf{v} \leq$$

$$\leq \frac{3}{2} C_q \beta_l [\|f_m(t)\|_{l+\lambda-\theta} \|f_m(t)\|_\theta + \|f_m(t)\|_{l-\theta} \|f_m(t)\|_{\lambda+\theta} , \tag{3.16}$$

valid for some $\beta_l > 0$ and for any $0 \leq \theta \leq 2$. Inequality (3.16) follows (see, e.g., [2]) from an elementary inequality due to Povzner, [23], and will be also called Povzner inequality[2].

One can prove that f_m converges to a solution of (3.1), under a stronger condition on f_0 than in Prop. 3.1. Indeed, one has ([2])

[2] Povzner-like inequalities can be also proved for the models presented in the previous sections.

PROPOSITION **3.2** *If* $\|f_0\|_l < \infty$ *for some* $l \geq 4$*, then there exists a unique solution* $f \geq 0$ *of problem* (3.1) *such that* $(1 + |\mathbf{v}|^l)f(t) \in L^1$*. Moreover,* $\|f(t)\|_2 = \|f_0\|_2$ *(* $t \geq 0$*), and for any* $t_* > 0$*, there is some number* $K = K(t_*, \|f_0\|_2, l)$ *such that* $\|f(t)\|_l \leq K \|f_0\|_l \quad (0 \leq t \leq t_*)$*.*

Proof. Consider the equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}f + hf = Q_m^a(f),\ f(0) = f_0\ (t \geq 0), \tag{3.17}$$

where $h(\mathbf{v}) := C(1 + |\mathbf{v}|^2)\|f_0(\mathbf{v})\|_2$ and $Q_m^a(f) := Q_m + hf$.

If f_m is as in Prop. 3.1, but f_0 is as in Prop. 3.2, then f_m is also the unique positive solution of Eq. (3.17), which satisfies (3.14). Further, consider

$$\frac{\mathrm{d}}{\mathrm{d}t}f + hf = Q_m^b(f),\ f(0) = f_0\ (t \geq 0), \tag{3.18}$$

where $Q_m^b(f) := Q_m^+(f) - Q^-(f) + hf$.

Let $V(t) := \exp(-th)$. One can introduce recurrences similar to (3.10),

$$\widetilde{f}_m^{1,i} = 0, \quad \widetilde{f}_m^{n+1,i} = V(t)f_0 + \int_0^t V(t-s)Q_m^i(\widetilde{f}_m^{n,i})(s)\mathrm{d}s \quad (n \geq 1); \quad i = a, b. \tag{3.19}$$

Under condition (3.2), if $C > 0$ is sufficiently large, the operators Q_m^i are positive and isotone so that the sequences $\left\{\widetilde{f_m}^{n,i}(t)\right\}_n$ are positive and increasing $(i = a, b)$. Moreover, if $0 \leq (1 + |\mathbf{v}|^2)g \in L^1$, then $Q_m^a\ (g) \geq Q_m^b(g)$ and $Q_m^b\ (g) \geq Q_j^b(g)$ for all m, $0 \leq j \leq m$. Using the above properties, one finds by induction that

$$0 \leq \widetilde{f_j}^{n,b}(t) \leq \widetilde{f_m}^{n,b}(t) \leq \widetilde{f}_m^{n,a}(t) \leq f_m(t); \quad 0 \leq j \leq m. \tag{3.20}$$

Hence, the increasing sequences $\left\{\widetilde{f_m}^{n,i}(t)\right\}_n$ are convergent. Note that if we set $f_m^b(t) := \lim_{n\to\infty} \widetilde{f_m}^{n,b}(t)$, then $0 \leq f_j^b(t) \leq f_m^b(t) \leq f_m(t)$; $0 \leq j \leq m$. Then $\left\{f_m^b(t)\right\}_n$ is increasing and $\left\|f_m^b(t)\right\|_2 \leq \|f_0\|_2$, hence $\left\{f_m^b(t)\right\}_n$ converges to some limit $f(t)$, as $m \to \infty$, and

$$\|f(t)\|_2 \leq \|f_0\|_2\,. \tag{3.21}$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}f + hf = Q(f) + hf \tag{3.22}$$

and, by (3.15),

$$\|f(t)\|_l \leq K \|f_0\|_l \quad (0 \leq t \leq t_*), \quad l \geq 4. \tag{3.23}$$

Thus f is a solution of (3.1) if there is equality in (3.21). This can be proved by estimating $s_m := f_m - f_m^b(t)$. Indeed, as f_m is the solution of (3.17), (3.18), one can write

$$\frac{\mathrm{d}}{\mathrm{d}t} s_m + h s_m = Q_m^a(f_m) - Q_m^b(f_m^b). \tag{3.24}$$

A short computation, which takes advantage that s_m is non-negative, and applies (3.23), gives (under hypothesis (3.2))

$$\|s_m(t)\|_2 \leq tCK \|f_0\|_4 \sup_{0 \leq s \leq t_*} \|s_m(s)\|_2 + o(1) \tag{3.25}$$

as $m \to \infty$ (with $C > 0$ sufficiently large, and K, t_* as in (3.23)).

Then for t sufficiently small, $\|s_m(t)\|_2 \to 0$ as $m \to \infty$, hence $\|f(t)\|_2 = \lim_{m\to\infty} \|f_m^b(t)\|_2 = \lim_{m\to\infty} \|f_m(t)\|_2 = \|f_0\|_2$.

To prove the uniqueness part of the proposition, observe that if $g \geq 0$ satisfies Eq. (3.1), and if $\|g(t)\|_2 \leq \infty$, then $\|g(t)\|_2 = \|f_0\|_2$. But g also satisfies the mild form of (3.22). Then $g \geq f$, by the construction of f. □

Variants of Arkeryd's monotonicity argument were successfully applied to other models close to the classical Boltzmann equation, [18], [27], [9], [7]. Thus, developing the above line of reasoning within a more general framework has become a tempting task. But this is not trivial, and requires new ideas (as will be seen in this section). Indeed, for instance, too key issues of Arkeryd's analysis seem rather specific to the model considered in [2]: a) choice of a priori estimates; b) construction of suitable regular operator approximations of the Boltzmann collision operators.

3.2. An abstract model

We begin with some terminology and facts related to Banach lattices ([17, 24]).

The frame of our analysis is a separable AL-space X with norm $\|\cdot\|$, order $\leq$, and positive cone X_+. We recall that an (AL) space, is a Banach lattice whose norm satisfies

$$\|g + h\| = \|g\| + \|h\| \quad (g, h \in X_+). \tag{3.26}$$

As X is an AL-space, if $h : \mathbb{R} \mapsto X_+$ is Bochner integrable, then property (3.26) gives

$$\left\| \int_{\mathcal{S}} h(s)\mathrm{d}s \right\| = \int_{\mathcal{S}} \|h(s)\| \, \mathrm{d}s \tag{3.27}$$

for any measurable set $\mathcal{S}$ of $\mathbb{R}$, the integral being in the sense of Lebesgue.

Examples of AL-spaces are L^1-real and the real subspace of self-adjoint trace-class operators (with trace norm)[3].

Related to the order of X, we shall also use the standard notations $(g \geq h) \Leftrightarrow (h \leq g)$, as well as $(g < h) \Leftrightarrow (h > g) \Leftrightarrow (g \leq h$ and $g \neq h)$. AL-spaces are *monotone complete*, in the sense that any increasing (i.e., directed $\leq$) norm-bounded family converges. The norm of an AL-space is *order continuous*, i.e., any directed $\geq$ filters that converges to 0 is also norm convergent to 0 . A map $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$, with $\mathcal{D}(\Gamma) \cap X_+ \neq \emptyset$, is called *positive (strictly positive)* if $0 \leq \Gamma g$ for $0 \leq g \in \mathcal{D}(\Gamma)$ (if $0 < \Gamma g$ for $0 < g \in \mathcal{D}(\Gamma)$). Further, $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ is called *isotone (strictly isotone)* if $\Gamma g \leq \Gamma h$, whenever $g \leq h$ (if $\Gamma g < \Gamma h$, whenever $g < h$), $g, h \in \mathcal{D}(\Gamma)$. Obviously, if $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ is isotone, $0 \in \mathcal{D}(\Gamma)$ and $0 \leq \Gamma(0)$, then Γ is positive. We say that a subset $\mathcal{M} \subset X$ is *p-saturated* (positively saturated) if $\mathcal{M} \cap X_+ \neq \emptyset$, and from $0 \leq g \leq h \in \mathcal{M}$, it follows that $g \in \mathcal{M}$. An operator $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ will be called *o-closed* (closed with respect to the order) if for any increasing sequence $\{g_n\} \subset \mathcal{D}(\Gamma)$ such that $\{g_n\}$ is increasing and convergent (in symbols, $\nearrow$) to some g, and $\{\Gamma g_n\}$ is Cauchy, one has $g \in \mathcal{D}(\Gamma)$ and $\lim_{n\to\infty} \Gamma g_n = \Gamma g$. Clearly, any closed mapping is o-closed.

We recall (see, e.g., [16]) that if $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ is a closed linear operator, then

$$\Gamma \int_{\mathbb{S}} h(s)\mathrm{d}s = \int_{\mathbb{S}} \Gamma h(s)\mathrm{d}s. \tag{3.28}$$

for any function h Bochner integrable on some measurable set $\mathbb{S} \in \mathbb{R}$, with values in $\mathcal{D}(\Gamma)$, and such that Γh is Bochner integrable.

We recall that a *positive* C_0 *semigroup* on X is a C_0 semigroup of positive linear operators on X. If $\left\{S^t\right\}_{t\geq 0}$ is a positive C_0 semigroup on X, then its infinitesimal generator G is densely defined and closed (as the infinitesimal generator of a C_0 semigroup). Moreover, G^k is densely defined and closed, $k = 2, 3, ...$. Additional useful properties are collected in the following lemma.

Let I denote the identity on X. Set $\mathcal{D}_+^\infty(G) := \cap_{k=1}^\infty \mathcal{D}(G^k) \cap X_+$.

[3] Actually, according to Kakutani's theorem, [24], every AL-space is isometrically isomorphic (as an ordered vector space) to a space of type L^1.

LEMMA **3.1** ([11])

a) *The sets* $\mathcal{D}(G^k) \cap X_+$, $k = 1, 2, ...,$ *and* $\mathcal{D}_+^\infty(G)$ *are dense in* X_+.

b) *Suppose that there is some number* $\gamma \geq 0$ *such that*

$$(G + \gamma I)g \leq 0 \quad (g \in \mathcal{D}(G) \cap X_+). \tag{3.29}$$

Then $\mathcal{D}(G^k) \cap X_+$, $k = 1, 2, ...,$ *and* $\mathcal{D}_+^\infty(G)$ *are p-saturated. Moreover, for any* $h \in X_+$,

$$0 \leq S^t h \leq \exp(-\gamma t)h \quad (t \geq 0), \tag{3.30}$$

and there is an increasing sequence $\{h_n\} \subset \mathcal{D}_+^\infty$, *such that* $h_n \nearrow h$ *as* $n \to \infty$.

Motivated by the examples of the previous section, it is of interest to consider the following abstract i.v.p., [11],

$$\frac{\mathrm{d}f}{\mathrm{d}t} = Q(t, f) = Q^+(t, f) - Q^-(t, f), \quad f(0) = f_0 \in X_+ \quad (t > 0), \tag{3.31}$$

formulated in X_+ (the particular autonomous case is not excluded).

In Eq. (3.31), Q^+ and Q^- are mappings defined from $\mathbb{R}_+ \times \mathcal{D}$ to X, for some $\mathcal{D} \subset X$ such that $\mathcal{D} \cap X_+$ is dense in X_+.

The following properties are assumed for $Q^\pm$:

a) For a.e. $t \geq 0$, the operators $Q^\pm(t, \cdot) : \mathcal{D} \mapsto X$ are positive and isotone.

b) The mappings $\mathbb{R}_+ \ni t \mapsto Q^\pm(t, g(t)) \in X_+$ are measurable for any Lebesgue measurable function $g : \mathbb{R}_+ \mapsto X$ that satisfies $g(t) \in \mathcal{D} \cap X_+$ a.e. on $\mathbb{R}_+$.

c) For a.e. $t \geq 0$, the operators $Q^\pm(t, \cdot)$ are o-closed and their common domain $\mathcal{D}$ is p-saturated.

We are interested in the existence and uniqueness of positive (i.e., in X_+) strong solutions of Eq. (3.31) under additional hypotheses which abstract further properties of the Boltzmann model.

We recall that a function $f : \mathbb{R}_+ \mapsto X$ is a strong solution of Eq. (3.31), if it is absolutely continuous on $\mathbb{R}_+$, differentiable a.e. on $\mathbb{R}_+$, satisfies Eq. (3.31) a.e. on $\mathbb{R}_+$, and verifies the initial condition. Equivalently, f is a strong solution of problem (3.31) if it is solution of the integral equation

$$f(t) = f_0 + \int_0^t Q(s, f(s))\mathrm{d}s \quad (t \geq 0), \tag{3.32}$$

where the integral is in the sense of Bochner.

We also consider the following problem related to Eq. (3.31)

$$\frac{\mathrm{d}f}{\mathrm{d}t} = Af + Q(t,f), \quad f(0) = f_0 \in X_+ \quad (t > 0), \tag{3.33}$$

with Q as in Eq. (3.31). Here A is the infinitesimal generator of a C_0 group of positive linear isometries on X, which commutes with Λ.

We are interested in the existence and uniqueness of mild solutions of Eq. (3.31) in X_+, i.e, solutions of the integral equation

$$f(t) = U^t f_0 + \int_0^t U^{t-s} Q(s, f(s))\mathrm{d}s \quad (t \geq 0) \tag{3.34}$$

in X_+, where $\{U^t\}_{t\in\mathbb{R}}$ is the C_0 group of positive linear isometries on X, generated by A (the integral is in the sense of Bochner).

As the above model is still too general for developing an existence theory of solutions, additional hypotheses are needed. The examples of the previous section suggest to assume some sort of dissipation (conservation) property, [11]. This claims the existence of a positive, densely defined, closed linear operator $\Lambda : \mathcal{D}(\Lambda) \subset X \mapsto X$ such that, for any positive solution $f(t) \in \mathcal{D}(\Lambda^2)$ of Eq. (3.31), the quantity $\|\Lambda f(t)\|$ is dissipated (conserved), i.e., is decreasing (constant) in t, and $\left\|\Lambda^2 f(t)\right\|$ is locally bounded in t. The "law of decrease" of $\|\Lambda f(t)\|$ can be used as a "natural" a priori estimate[4]. In particular,

$$\|\Lambda f(t)\| \leq \|\Lambda f_0\| \quad (t \geq 0). \tag{3.35}$$

To be precise, we introduce the following "dissipation" property ([11]). Let $\mathcal{M}$ be a subset of $\mathcal{D} \cap X_+$ dense in X_+.

Definition **3.1** ([11]) *A closed positive linear operator* $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ *is called of type D on* $\mathcal{M}$ *(with respect to Eq. (3.31)) if* $\mathcal{M} \subset \mathcal{D}(\Gamma)$, $Q^{\pm}(t,\mathcal{M}) \subset \mathcal{D}(\Gamma)$ *a.e. on* $\mathbb{R}_+$*, and for any* $g \in \mathcal{M}$,

$$0 \leq \Delta(t, g; \Gamma, Q) := \left\|\Gamma Q^-(t,g)\right\| - \left\|\Gamma Q^+(t,g)\right\| \quad (t \geq 0 \quad a.e.). \tag{3.36}$$

If Γ is of type D on $\mathcal{M}$, then the following property can be easily established by making use of (3.27) and (3.28).

Lemma **3.2** ([11]) *Let* g_0, $g(t)$, $h(t) \in \mathcal{M}$, $t \geq 0$ *a.e., with* $Q^{\pm}(\cdot, h(\cdot))$, $\Gamma Q^{\pm}(\cdot, h(\cdot)) \in L^1_{loc}(\mathbb{R}_+; X_+)$*, and*

$$g(t) \leq g_0 + \int_0^t Q(s, h(s))\mathrm{d}s \quad (t \geq 0). \tag{3.37}$$

[4]This can take various forms in applications, depending on the form of Λ and Q, e.g., conservation energy, in the case of the model of [2].

Then

$$\|\Gamma g(t)\| + \int_0^t \Delta(s, h(s); \Gamma, Q)\mathrm{d}s \leq \|\Gamma g_0\| \quad (t \geq 0). \tag{3.38}$$

Moreover, (3.38) *holds with equality sign for any* $t \geq 0$*, provided that there is equality in* (3.37) *for all* $t \geq 0$.

On the other hand, in determining the behavior of $\left\|\Lambda^2 f(t)\right\|$, a major role appears to be played by the Povzner inequality (3.16). This has to be somehow included in the model.

Now we are in position to complete the setting of Eq. (3.31) with additional hypotheses, making more precise the above considerations.

Specifically, we assume that there is a linear operator $\Lambda : \mathcal{D}(\Lambda) \subset X \mapsto X$, with $\mathcal{D}(\Lambda) \subset \mathcal{D}$ and $Q^{\pm}(t, \mathcal{D}(\Lambda^k) \cap X_+) \subset \mathcal{D}(\Lambda^{k-1})$, $t \geq 0$ a.e., $k = 2, 3$, such that:

(A_0) The operator $(-\Lambda)$ is the infinitesimal generator of a C_0 semigroup of positive linear operators on X, and there is a number $\lambda_0 > 0$ such that

$$(\Lambda - \lambda_0 I)g \geq 0 \quad (g \in \mathcal{D}(\Lambda) \cap X_+). \tag{3.39}$$

(A_1) For a.e. $t \geq 0$,

$$\Delta(t, g) := \Delta(t, g; \Lambda, Q) \geq 0 \quad (g \in \mathcal{D}(\Lambda^2) \cap X_+), \tag{3.40}$$

and the map $\mathcal{D}(\Lambda^2) \cap X_+ \ni g \mapsto \Delta(t, g) \in \mathbb{R}_+$ is isotone.

(A_2) There exists a non-decreasing convex function $a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

$$a(\|\Lambda g\|)\Lambda g - Q^-(t, g) \geq 0, \quad (g \in \mathcal{D}(\Lambda) \cap X_+, \quad t \geq a.e.), \tag{3.41}$$

and for a.e. $t \geq 0$, the map $\mathcal{D}(\Lambda) \cap X_+ \ni g \mapsto a(\|\Lambda g\|)\Lambda g - Q^-(t, g) \in X$ is isotone.

(A_3) There exists a non-decreasing function $\rho : \mathbb{R}_+ \mapsto \mathbb{R}_+$, and there is an operator $\Lambda_1 : \mathcal{D}(\Lambda_1) \subset X \mapsto X$ of type D on $\mathcal{D}(\Lambda^2) \cap X_+$ such that

$$-\Delta(t, g; \Lambda^2, Q) \leq \rho(\|\Lambda_1 g\|) \left\|\Lambda^2 g\right\| \quad (g \in \mathcal{D}(\Lambda^3) \cap X_+, \, t \geq 0 \; a.e.). \tag{3.42}$$

Some remarks are in order.

First, observe that if $g \in \mathcal{D}(\Lambda^2) \cap X_+$, then by (3.39), (3.40) and (3.41) we have the simple inequalities

$$\|g\| \leq \lambda_0^{-1} \|\Lambda g\| \leq \lambda_0^{-2} \left\|\Lambda^2 g\right\| \tag{3.43}$$

and

$$\left\|Q^{\pm}(t,g)\right\| \leq \lambda_0^{-1}\left\|\Lambda Q^{\pm}(t,g)\right\| \leq \lambda_0^{-1}\left\|\Lambda Q^{-}(t,g)\right\| \leq$$

$$\leq a(\|\Lambda g\|)\lambda_0^{-1}\left\|\Lambda^2 g\right\| \leq a(\lambda_0^{-1}\left\|\Lambda^2 g\right\|)\lambda_0^{-1}\left\|\Lambda^2 g\right\| \quad (t \geq 0 \quad a.e.), \qquad (3.44)$$

with the following obvious consequences.

Remark **3.1** *$Q^{\pm}(t,0)=0$ and $\Delta(t,0)=0$ a.e. on $\mathbb{R}_+$.*

Let $\Lambda^0 := I$.

Remark **3.2** *If $g : \mathbb{R}_+ \mapsto X_+$ is measurable, with $g(t) \in \mathcal{D}(\Lambda^2)$, $t \geq 0$, a.e., and $\left\|\Lambda^2 g\right\| \in L^{\infty}_{loc}(\mathbb{R}_+)$, then g, $\Lambda^{k+1}g$, and $\Lambda^k Q^{\pm}(\cdot, g(\cdot))$ are in $L^1_{loc}(\mathbb{R}_+; X_+)$, $k=0,1$.*

Lemma 3.1a) and (A_0) imply that $\mathcal{D}(\Lambda^k) \cap X_+$, $k = 1,2,...$, and $\mathcal{D}^{\infty}_{+} := \mathcal{D}^{\infty}_{+}(\Lambda)$ are p-saturated and dense in X_+. Obviously, (3.39) shows that Λ is positive. Thus, by (3.40), the operator Λ is of type D on $\mathcal{D}(\Lambda^2) \cap X_+$. This has the following important consequence.

If $f(t) \in \mathcal{D}(\Lambda^2)$, $t \geq 0$, a.e., and if $Q^{\pm}(\cdot, f(\cdot))$, $\Lambda Q^{\pm}(\cdot, f(\cdot)) \in L^1(\mathbb{R}_+; X_+)$, then by (3.38), applied with equality sign,

$$\|\Lambda f(t)\| + \int_0^t \Delta(s, f(s))\mathrm{d}s = \|\Lambda f_0\| \quad (t \geq 0). \qquad (3.45)$$

Thus $\|\Lambda f(t)\|$ is decreasing in time and satisfies (3.35). In particular, if $\Delta(t,g)=0$ for all $g \in \mathcal{D}(\Lambda^2) \cap X_+$, $t \geq 0$ a.e., then $\|\Lambda f(t)\|$ is conserved for all $t \geq 0$.

Observe that inequality (3.42) is of the form

$$-\Delta(t,g;\Gamma,Q) \leq \rho_{\Gamma}(\|\Lambda_1 g\|)\,\|\Gamma g\| \quad (g \in \mathcal{M}_1,\, t \geq 0 \; a.e.), \qquad (3.46)$$

where $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ is some positive linear operator, and $\mathcal{M}_1 \subset \mathcal{D}(\Gamma) \cap \mathcal{D}(\Lambda^2) \cap X_+$ is such that $Q^{\pm}(t, \mathcal{M}_1) \subset \mathcal{D}(\Gamma)$, $t \geq 0$ a.e., while $\rho_{\Gamma} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is some non-decreasing function.

Formula (3.45) generalizes a priori estimates introduced in e.g., [2, 7, 8, 9, 27]. Formula (3.46) can be regarded as an abstract correspondent to the Povzner inequality, [2, 23].

We finally remark that the above setting does not exclude the case $\Lambda_1 = \Lambda$ when, obviously, some of the above conditions become redundant.

3.3. General results on the existence of solutions

We are now in position to state some results ([11], [13]) on the existence of solutions to our abstract model. The proofs will be sketches in the next subsection (for more details, the reader is referred to [11] and [13]). First we consider problem (3.31).

THEOREM **3.1** *Let either of the following two sets of conditions be fulfilled:*
a) $Q^+(t, \mathcal{D}_+^\infty) \subset \mathcal{D}_+^\infty$, $t \geq 0$ *a.e.*, $\Lambda^k Q^+(\cdot, \mathcal{D}_+^\infty) \subset L^1_{loc}(\mathbb{R}_+; X_+)$, $k = 1, 2, \ldots$. *In problem* (3.31), $f_0 \in \mathcal{D}(\Lambda^2) \cap X_+$.
b) *The operators* $Q^\pm$ *do not depend explicitly on* t. *In problem* (3.31), $f_0 \in \mathcal{D}(\Lambda^3) \cap X_+$.
Then there exists a unique positive strong solution of the i.v.p. (3.31) *such that* $f(t) \in \mathcal{D}(\Lambda^2)$ *for any* $t \geq 0$, *and* $\left\|\Lambda^2 f(\cdot)\right\|$ *is locally bounded on* $\mathbb{R}_+$.
Moreover, $f, \Lambda f \in C(\mathbb{R}_+; X_+)$. *Furthermore,* f *satisfies Eq.* (3.45) *and*

$$\left\|\Lambda^2 f(t)\right\| \leq \exp(\rho(\|\Lambda_1 f_0\|)t) \left\|\Lambda^2 f_0\right\| \quad (t \geq 0). \tag{3.47}$$

Note here that Theorem 3.1a) is also applicable to the autonomous case, but, clearly, its conditions are different from those of Theorem 3.1b).

Theorem 3.1 has an immediate noticeable consequence, as follows:

Consider Eq. (4.22) and let $\left\{U^t\right\}_{t\in\mathbb{R}}$ be the C_0 group of positive linear isometries on X, generated by A.

If f is a solution of (3.34), then setting $F(t) := U^{-t} f(t)$ in (3.34), we get

$$F(t) = f_0 + \int_0^t Q_U(s, F(s))\mathrm{d}s \quad (t \geq 0), \tag{3.48}$$

hence, by differentiation,

$$\frac{\mathrm{d}}{\mathrm{d}t}F = Q_U(t, F) = Q_U^+(t, F) - Q_U^-(t, F), \quad F(0) = f_0 \quad (t \geq 0 \quad a.e.), \tag{3.49}$$

where $Q_U(t, \cdot) := U^{-t} Q(t, U^t \cdot)$ and $Q_U^\pm(t, \cdot) := U^{-t} Q^\pm(t, U^t \cdot)$.

Suppose that $U^t \mathcal{D}(\Lambda) = \mathcal{D}(\Lambda)$ and $U^t \Lambda = \Lambda U^t$ on $\mathcal{D}(\Lambda)$ for every $t > 0$. Also, let $U^t \mathcal{D}(\Lambda_1) = \mathcal{D}(\Lambda_1)$ and $U^t \Lambda_1 = \Lambda_1 U^t$ on $\mathcal{D}(\Lambda_1)$ for all $t > 0$.

Now $Q_U^\pm$ and Q_U are well defined as maps from $\mathbb{R}_+ \times \mathcal{D}(\Lambda)$ to X, the last equation is of the form (3.31), and we can state the following consequence ([11]) of Theorem 3.1a):

Corollary **3.1** *Let $Q^+(t,\mathcal{D}_+^\infty) \subset \mathcal{D}_+^\infty$, $t \geq 0$ a.e., and $\Lambda^k Q^+(\cdot, U^{\cdot} g) \in L^1_{loc}(\mathbb{R}_+; X_+)$ for all $g \in \mathcal{D}_+^\infty$, $k = 1, 2, \ldots$. Suppose that $f_0 \in \mathcal{D}(\Lambda^2) \cap X_+$ in* (4.22). *Then problem* (4.22) *has a unique positive mild solution f such that $f(t) \in \mathcal{D}(\Lambda^2)$ for any $t \geq 0$ and $\|\Lambda^2 f(\cdot)\|$ is locally bounded on $\mathbb{R}_+$. Moreover, $f, \Lambda f \in C(\mathbb{R}_+; X_+)$. Furthermore, f satisfies* (3.45) *and* (3.47).

The following result, [13], extends the existence of strong solutions of Eq. (3.31) to the case of initial datum $f_0 \in \mathcal{D}(\Lambda) \cap X_+$ (instead of $\mathcal{D}(\Lambda^2) \cap X_+$, as assumed in Theorem 3.1).

Theorem **3.2** *Under the assumptions of Theorem* 3.1a) *on Λ and $Q^\pm$, let $f_0 \in \mathcal{D}(\Lambda) \cap X_+$ in Eq.* (3.31). *Then there exists a strong solution, $f \in C([0,\infty); X_+)$, of the i.v.p.* (3.31). *Moreover, for any $t \geq 0$, $f(t) \in \mathcal{D}(\Lambda)$, $\|\Lambda f(t)\| \leq \|\Lambda f_0\|$, and*

$$\|f(t)\| = \|f_0\| + \int_0^t \left\|Q^+(s, f(s))\right\| - \left\|Q^-(s, f(s))\right\| \mathrm{d}s. \tag{3.50}$$

Note here that if f is as in Theorem 3.2, we know only that $f \in \mathcal{D}(\Lambda) \cap X_+$. Then $\Delta(t,f)$ and $\Lambda^2 f$ may not be not well-defined. Therefore, we cannot obtain inequalities of the form (3.45) (except the case when $\Delta = 0$ on $\mathcal{D}(\Lambda^2) \cap X_+$,) or like (3.47), at the level of abstraction of the theorem.

Also remark that Theorem 3.2 leaves open the question on the uniqueness of the solution in the general case (under the conditions of the theorem).

However, uniqueness can be proved under additional conditions, [13].

Proposition **3.3** *If $\Delta(t,g) = 0$ for all $g \in \mathcal{D}(\Lambda^2) \cap X_+$, t – a.e., then*

$$\|\Lambda f(t)\| = \|\Lambda f_0\| \quad (t \geq 0), \tag{3.51}$$

and there is a unique solution of the i.v.p. (3.31) *as in Theorem* 3.2, *which satisfies* (3.51).

A similar result like Corollary 3.1 can be formulated for Theorem 3.2.

The following proposition yields additional useful estimates, [11], for the solutions of Eq. (3.31). For simplicity, we remain in the conditions of Theorem 3.1a). However, similar results are valid when Theorem 3.1b) holds, as can be seen by inspecting the proof of the proposition.

Assume that $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ is a closed, positive linear operator. Let f be a solution of problem (3.31), provided by Theorem 3.1a).

PROPOSITION **3.4** a) *Suppose that* Γ *is of type D on* $\mathcal{D}_+^\infty$. *Then* $f(t) \in \mathcal{D}(\Gamma)$, $t \geq 0$, *and*

$$\|\Gamma f(t)\| \leq \|\Gamma f_0\| \quad (t \geq 0). \tag{3.52}$$

b) *Suppose that* Γ *and* ρ_Γ *are as in* (3.46), *with* $\mathcal{M}_1 \supseteq \mathcal{D}_+^\infty$. *Then* $f(t) \in \mathcal{D}(\Gamma)$, $t \geq 0$, *and*

$$\|\Gamma f(t)\| \leq \exp(\rho_\Gamma(\|\Lambda_1 f_0\|)t)\ \|\Gamma f_0\| \quad (t \geq 0). \tag{3.53}$$

In applications, the choice of Λ and Λ_1 may be not unique. In some cases, the role of Λ_1 and Γ may be played by suitable powers of Λ, while, in other examples, $\Lambda = \Lambda_1 = \Gamma$.

A correspondent to Prop. 3.4, applicable to Corollary 3.1, can be readily obtained. The modifications in the reformulation of the proposition are obvious and include additional hypotheses for the commutation of U^t with Γ, etc.

3.4. Proofs

Sketch of the proof of Theorem 3.1

In the following, we give an insight into the rather lengthy argument of Theorem 3.1 (see [11] for a detailed proof), and explain the role of assumptions (A_0)-(A_3).

We start by observing that if $f_0 = 0$ in (3.31), then, by Remark 3.1, clearly $f(t) \equiv 0$ is a solution to Eq. (3.31). It is the unique strong solution in $\mathcal{D}(\Lambda^2) \cap X_+$, as it follows from (3.45). Moreover, if $0 \neq f_0 \in \mathcal{D}(\Lambda^2) \cap X_+$, but $a(\|\Lambda f_0\|) = 0$, then $Q^\pm(t, f_0) = 0$, for a.e. $t \geq 0$, by (3.44), hence $f(t) \equiv f_0$ is a solution to (3.31). It is the unique solution in $\mathcal{D}(\Lambda^2) \cap X_+$, because any other solution $f^*(t) \in \mathcal{D}(\Lambda^2) \cap X_+$ must be a.e. constant. Indeed, applying (3.45), and invoking the positivity and monotonicity of a, we obtain $0 \leq a(\|\Lambda f^*(t)\|) \leq a(\|\Lambda f_0\|) = 0$. This leads (again by (3.44)) to $Q^\pm(t, f(t)) = 0$ a.e.

Therefore, one can assume below that $f_0 \neq 0$ and $a(\|\Lambda f_0\|) \neq 0$.

We first refer to the **existence** part of the theorem. Inspired from [2], one can consider the problem

$$\frac{\mathrm{d}}{\mathrm{d}t} f + a(\|\Lambda f_0\|)\Lambda f = B(t, f, f), \quad f(0) = f_0 \in X_+ \quad (t \geq 0). \tag{3.54}$$

Here a is as in (A_2), and B is formally defined by

$$B(t,g,h) := Q(t,g(t)) + a\left(\|\Lambda g(t)\| + \int_0^t \Delta(s,h(s))\mathrm{d}s\right)\Lambda g(t) \quad (t \geq 0 \quad a.e.) \tag{3.55}$$

for all $g(t) \in \mathcal{D}(\Lambda) \cap X_+$ and $h(t) \in \mathcal{D}(\Lambda^2) \cap X_+$ with $\Lambda Q^{\pm}(\cdot, h(\cdot)) \in L^1_{loc}(\mathbb{R}_+; X_+)$.

By (3.45), any strong positive solution of Eq. (3.31) is also a solution to (3.54). Conversely, any positive strong solution of problem (3.54) is a solution of Eq. (3.31), provided that it satisfies (3.45).

Recall now that, by (A_0) and Lemma 3.1b), the operator $L = -a(\|\Lambda f_0\|)\Lambda$ is the infinitesimal generator of a C_0 positive semigroup $\{V^t\}_{t\geq 0}$, and

$$0 \leq V^t h \leq \exp(-a(\|\Lambda f_0\|)\lambda_0 t)h \leq h \quad (h \in X_+). \tag{3.56}$$

Thus any solution of Eq. (3.54) is also a solution of the mild problem

$$f(t) = V^t f_0 + \int_0^t V^{t-s} B(s,f,f)\mathrm{d}s\,, \tag{3.57}$$

the integral being in the sense of Bochner.

Eq. (3.57) is useful for monotone iteration. Indeed, $\{V^t\}_{t\geq 0}$ is positive, and one can prove[5] the following properties ([11]).

Lemma **3.3** *Let g_i, h_i, $i = 1,2$, satisfy the conditions of Remark 3.2. Suppose that $g_1(t) \leq g_2(t)$ and $h_1(t) \leq h_2(t)$ a.e. on $\mathbb{R}_+$. Then $B(\cdot, g_i, h_j) \in L^1_{loc}(\mathbb{R}_+; X_+)$, $i,j = 1,2$. In addition, for a.e. $t \geq 0$,*

$$0 \leq B(t, g_1, h_1) \leq B(t, g_2, h_2). \tag{3.58}$$

Thus, formally, by (3.57) one could consider the following iteration, hopefully, increasing:

$$f_1(t) = 0, \quad f_2(t) = V^t f_0, \tag{3.59}$$

$$f_n(t) = V^t f_0 + \int_0^t V^{t-s} B(s, f_{n-1}, f_{n-2})\mathrm{d}s \quad (n = 3,4,...). \tag{3.60}$$

Note that if $\{f_n(t)\}_n$ is sufficiently regular, by differentiation, (3.60) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} f_n(t) = B(t, f_{n-1}, f_{n-2}) - a(\|\Lambda f_0\|)\Lambda f_n(t) \quad (t > 0 \quad a.e., \quad n \geq 3), \tag{3.61}$$

[5]See the Appendix.

and integrating (3.61) one has

$$f_n(t) = f_0 + \int_0^t Q(s, f_{n-1}(s))\mathrm{d}s +$$

$$+ \int_0^t a\left(\|\Lambda f_{n-1}(s)\| + \int_0^s \Delta(\tau, f_{n-2}(\tau))\mathrm{d}\tau\right)\Lambda f_{n-1}(s)\mathrm{d}s.$$

$$- \int_0^t a(\|\Lambda f_0\|)\Lambda f_n(s)\mathrm{d}s. \qquad (3.62)$$

However, in general, $B(\cdot, g, h)$ does not exist for all g, $h \in X$. Hence we need give a meaning to (3.60), at least for f_0 in a sufficiently large set. Here comes the role of $\mathcal{D}_+^\infty$ (of $\mathcal{D}(\Lambda^3) \cap X_+$). Indeed, if $f_0 \in \mathcal{D}_+^\infty$ ($f_0 \in \mathcal{D}(\Lambda^3) \cap X_+$), then one can show that $f_n(t) \in \mathcal{D}_+^\infty$ ($f_0 \in \mathcal{D}(\Lambda^3) \cap X_+$), and is sufficiently regular. This is clarified in the lemma bellow, which summarizes the main results[6] of [11] on the properties of $\{f_n(t)\}_n$.

LEMMA **3.4** a) *In addition, to the conditions of Theorem* 3.1a), *let* $f_0 \in \mathcal{D}_+^\infty$. *Then* $f_n(t)$, $Q^\pm(t, f_n(t)) \in \mathcal{D}_+^\infty$ *a.e. on* $\mathbb{R}_+$. *Moreover,* $\Lambda^k Q^\pm(\cdot, f_n(\cdot)) \in L^1_{loc}(\mathbb{R}_+; X_+)$, $k = 0, 1, ...$, $n = 1, 2, ...$.

b) *Assume the conditions of Theorem* 3.1b). *Then* $f_n(t) \in \mathcal{D}(\Lambda^3) \cap X_+$ *and* $Q^\pm(f_n(t)) \in \mathcal{D}(\Lambda^2) \cap X_+$; $t \geq 0$. *Moreover,* $\Lambda^k Q^\pm(f_n) \in L^1_{loc}(\mathbb{R}_+; X_+)$, $k = 0, 1, 2, \ , n = 1, 2, ...$.

c) *In both cases* a) *and* b), $\Lambda^k f_n \in C(\mathbb{R}_+; X_+)$, $k = 0, 1, 2$, *and* f_n *is a.e. differentiable on* $\mathbb{R}_+$ *and satisfies* (3.61) *(and* (3.62)*). Moreover, for any* $t \geq 0$, *the sequence* $\{f_n(t)\}_n$ *is increasing.*

d) *If* $f_n(t)$ *is as in* a) *or* b), *and* $n \geq 2$, *then*

$$f_n(t) \leq f_0 + \int_0^t Q(s, f_{n-1}(s))\mathrm{d}s \qquad (3.63)$$

and

$$\|\Lambda f_n(t)\| + \int_0^t \Delta(s, f_{n-1}(s))\mathrm{d}s \leq \|\Lambda f_0\|. \qquad (3.64)$$

e) *If* $f_n(t)$ *is as in* a) *or* b), *and* Γ *is an operator of type* D *on* $\mathcal{D}_+^\infty$, *(on* $\mathcal{D}(\Lambda^2) \cap X_+$*) then for any* $t \geq 0$,

$$\|\Gamma f_n(t)\| \leq \|\Gamma f_0\| \quad (n = 1, 2, ...). \qquad (3.65)$$

[6] See the Appendix for a proof.

In particular,

$$\left\|\Lambda^2 f_n(t)\right\| \le \exp(\rho(\|\Lambda_1 f_0\|)t) \left\|\Lambda^2 f_0\right\| \quad (t \ge 0, \quad n = 1, 2, ...), \tag{3.66}$$

with ρ *as in* (3.42).

f) *Suppose that* $f_n(t)$ *is as in* a) *(as in* b)*). Let* $\Gamma : \mathcal{D}(\Gamma) \subset X \mapsto X$ *be some closed, positive linear operator, satisfying* (3.46)*, with* $\mathcal{M}_1 \supseteq \mathcal{D}_+^\infty$ *(with* $\mathcal{M}_1 \supseteq \mathcal{D}(\Lambda^3) \cap X_+$*). Then for any* $t \ge 0$,

$$\|\Gamma f_n(t)\| \le \exp(\rho_\Gamma(\|\Lambda_1 f_0\|)t) \|\Gamma f_0\| \quad (n = 1, 2, ...), \tag{3.67}$$

with ρ_Γ *as in* (3.46).

By the above lemma, $\{f_n(t)\}_n$ is increasing, and the key inequality (3.64) shows that $\{f_n(t)\}_n$ is norm bounded[7]. Thus $\{f_n(t)\}_n$ is convergent, because X is monotone complete. One expects the limit to satisfy (3.54) (and (3.57), too). The proof hinges on the application of Lebesgue's dominated convergence theorem to (3.62) (as the operators $Q^\pm$ are o-closed, and Λ is closed). To this end, the limit of $\{f_n(t)\}_n$ must be in $\mathcal{D}(\Lambda^2)$, which follows from (3.66). Now, to prove that the limit of $\{f_n(t)\}_n$ is a strong solution to (3.31), it remains to show that the above limit satisfies (3.45). This is done by applying Gronwall's Lemma to an inequality to be obtained from (3.62) (by using (3.66) and the convexity of a). But the above procedure provides the existence part of the Theorem 3.1a) only for $f_0 \in \mathcal{D}_+^\infty$, hence one more step is needed. Since $\mathcal{D}_+^\infty$ is dense in X_+ (cf. Lemma 3.1), any initial datum as in the assumptions of Theorem 3.1a), can be approximated by elements of $\mathcal{D}_+^\infty$. This leads to a monotone scheme approximating (3.60) and one can apply successively Lebesgue's convergence theorem. In details, one proceeds as follows.

Step A. If in addition to the conditions of Theorem 3.1 a), one assumes $f_0 \in \mathcal{D}_+^\infty$ then Lemma 3.4 applies. As Λ^k is closed, clearly, by (3.39) and the monotone completeness of X, it follows that there is some $f(t) \in \mathcal{D}(\Lambda^k)$ such that $\Lambda^k f_n(t) \nearrow \Lambda^k f(t)$ as $n \to \infty$, $t \ge 0$, $k = 0, 1, 2$. Consequently, $f(t)$ satisfies (3.47). Moreover, Remark 3.2 implies that $\Lambda^k f$, $k = 0, 1, 2$, $Q^\pm(\cdot, f(\cdot))$, and $\Lambda Q^\pm(\cdot, f(\cdot))$ are in $L^1_{loc}(\mathbb{R}_+; X_+)$. Then, applying Lebesgue's dominated convergence theorem in (3.62) and (3.64), we get

$$f(t) = f_0 + \int_0^t Q(s, f(s))\mathrm{d}s +$$

[7] Inequality (3.64) motivates the construction (3.60) as a second-order recurrence. Indeed, except for the case $\Delta \equiv 0$, an inequality of the form (3.64) could not be proved if (3.60) was redefined with $B(s, f_{n-1}, f_{n-1})$ instead of $B(s, f_{n-1}, f_{n-2})$.

$$+\int_0^t \left[a\left(\|\Lambda f(s)\| + \int_0^s \Delta(\tau, f(\tau))\mathrm{d}\tau\right) - a(\|\Lambda f_0\|)\right] \Lambda f(s)\mathrm{d}s \quad (t \geq 0) \tag{3.68}$$

(i.e., f is a strong solution of Eq.(3.54)) and, also,

$$0 \leq \psi(t) := \|\Lambda f_0\| - \|\Lambda f(t)\| - \int_0^t \Delta(s, f(s))\mathrm{d}s \quad (t \geq 0). \tag{3.69}$$

Obviously, (3.68) implies $f, \Lambda f \in C(\mathbb{R}_+; X_+)$.

Note now the usefulness of (3.68): to prove that f is a strong solution of (3.31), it is sufficient to show that $\psi \equiv 0$ (which means exactly (3.45)).

To this end, first observe that since, by (A_2), a is non-decreasing and locally Lipschitz, then inequality (3.69) implies that there is a number $0 < c = c(\|\Lambda f_0\|)$, depending only on $\|\Lambda f_0\|$, such that

$$0 \leq a(\|\Lambda f_0\|) - a\left(\|\Lambda f(t)\| + \int_0^t \Delta(\tau, f(\tau))\mathrm{d}\tau\right) < c\psi(t). \tag{3.70}$$

Further rewriting Eq. (3.68) conveniently, and applying Λ to the resulting equation, one can invoke (3.26) and (3.27) to obtain

$$\psi(t) = \int_0^t \left[a(\|\Lambda f_0\|) - a\left(\|\Lambda f(s)\| + \int_0^s \Delta(\tau, f(\tau))\mathrm{d}\tau\right)\right] \left\|\Lambda^2 f(s)\right\| \mathrm{d}s. \tag{3.71}$$

As $f(t)$ satisfies (3.47), introducing (3.70) in (3.71), we find

$$0 \leq \psi(t) \leq c \int_0^t \psi(s) \left\|\Lambda^2 f(s)\right\| \mathrm{d}s \leq c_T \int_0^t \psi(s)\mathrm{d}s \quad (0 \leq t \leq T), \tag{3.72}$$

for each $T > 0$. Here, $c_T > 0$ is a number depending only on T and f_0.

Now the Gronwall inequality implies $\psi(t) = 0$, $0 \leq t \leq T$, for any $T > 0$. This concludes the existence part of the proof of the Theorem 3.1a), in the case $f_0 \in \mathcal{D}_+^\infty$).

Step B. We use the result of the previous step to prove the existence part of Theorem 3.1 a), in the case $f_0 \in \mathcal{D}(\Lambda^2) \cap X_+$, as follows. First note that by Lemma 3.1b), there is an increasing sequence $\{f_{0,i}\} \subset \mathcal{D}_+^\infty$ such that $f_{0,i} \nearrow f_0$, as $i \to \infty$. Then, by Step A, there is a sequence of strong solutions $\{F_i\}_i$ of Eq. (3.31) with $F_i(0) = f_{0,i}$, satisfying the properties of the theorem. In particular,

$$\left\|\Lambda^2 F_i(t)\right\| \leq \exp\left[\rho(\|\Lambda_1 f_{0,i}\|)\right] \left\|\Lambda^2 f_{0,i}\right\| \quad (t \geq 0). \tag{3.73}$$

In addition,

$$F_i(t) = f_{0,i} + \int_0^t Q(s, F_i(s))\mathrm{d}s, \tag{3.74}$$

$$\Lambda F_i(t) = \Lambda f_{0,i} + \int_0^t \Lambda Q(s, F_i(s))\mathrm{d}s, \tag{3.75}$$

and

$$\|\Lambda F_i(t)\| + \int_0^t \Delta(s, F_i(s))\mathrm{d}s = \|\Lambda f_{0,i}\|. \tag{3.76}$$

Moreover, by Step A, each F_i is the limit of an increasing sequence $\{f_{n,i}(t)\}_n$ defined by (3.60) with $f_{n,i}(0) = f_{0,i}$. But the positivity of V^t and Lemma 3.3 imply that if $f_{0,i} \leq f_{0,j}$, then $f_{n,i}(t) \leq f_{n,j}(t)$ for all n and $t \geq 0$. Then the sequence $\{F_i\}$ is increasing.

Furthermore, since $\|\Lambda_1 f_{0,i}\| \leq \|\Lambda_1 f_0\|$, $\left\|\Lambda^2 f_{0,i}\right\| \leq \left\|\Lambda^2 f_0\right\|$, and since ρ is non-decreasing, it follows from inequality (3.73) that

$$\left\|\Lambda^2 F_i(t)\right\| \leq \exp(\rho(\|\Lambda_1 f_0\|)t)\left\|\Lambda^2 f_0\right\| \quad (t \geq 0). \tag{3.77}$$

Now a convergence argument, as in the beginning of Step A, implies that there is an element $f \in L^1_{loc}(\mathbb{R}_+; X_+)$, with the properties stated in Remark 3.2, such that $F_i(t) \nearrow f(t)$ as $i \to \infty$, a.e. It remains to apply, say, Lebesgue's convergence theorem in (3.74)–(3.76) to conclude the existence part of Theorem 3.1a).

Existence in case b). In this case, Lemma 3.4 applies, corresponding to the fulfillment of the conditions of Theorem 3.1b). Then, the proof is as in Step A of case a).

Finally, we prove the **uniqueness** part of Theorem 3.1.

Let f be the solution of Eq. (3.31) provided by the existence part of this proof, and recall that it satisfies Eq. (3.45). If F is *another* positive solution of Eq. (3.31) with regularity properties as in Theorem 3.1, then F satisfies Eq. (3.45), too, hence

$$\|\Lambda f(t)\| + \int_0^t \Delta(s, f(s))\mathrm{d}s = \|\Lambda f_0\| = \|\Lambda F(t)\| + \int_0^t \Delta(s, F(s))\mathrm{d}s.$$

By Lebesgue's convergence theorem applied to (3.60), clearly, f also solves Eq. (3.57). On the other hand, F is a solution to (3.57). But $f \leq F$, because of the form of (3.60), so that

$$\|\Lambda f(t)\| + \int_0^t \Delta(s, f(s))\mathrm{d}s < \|\Lambda F(t)\| + \int_0^t \Delta(s, F(s))\mathrm{d}s$$

on some subset of $\mathbb{R}_+$ with nonzero Lebesgue measure. □

Proof of Theorem 3.2

As in the proof of Theorem 3.1, to exclude trivial situations, we suppose the $\|f_0\| \neq 0$ or $a(\|f_0\|) \neq 0$. By Lemma 3.1, there is a sequence $\{f_{n,0}\}_n \subset \mathcal{D}_+^\infty$ such that $f_{n,0} \nearrow f_0$ as $n \to \infty$. Then by Theorem 3.1a) the i.v.p. (3.31) with initial condition $f_{n,0}$ has a unique positive solutions $F_n \in \mathcal{D}(\Lambda^2) \cap X_+$ such that (3.31) provided by Theorem 3.1 with initial datum $f_{n,0}$ forms an increasing sequence such that $F_n, \Lambda F_n \in C(\mathbb{R}_+; X_+)$,

$$F_n(t) = f_{n,0} + \int_0^t Q^+(s, F_n(s))\mathrm{d}s - \int_0^t Q^-(s, F_n(s))\mathrm{d}s \quad (t \geq 0). \tag{3.78}$$

and

$$\|\Lambda F_n(t)\| + \int_0^t \Delta(s, F_n(s)\mathrm{d}s = \|\Lambda f_{n,0}\| \quad (t \geq 0). \tag{3.79}$$

But $\Delta(s, F_n(s) \geq 0$ so that

$$\|\Lambda F_n(t)\| \leq \|\Lambda f_{n,0}\| \leq \|\Lambda f_0\| \quad (t \geq 0). \tag{3.80}$$

Note now that F_n, $f_{n,0}$, $Q^\pm(t, F_n(t))$ are positive. Then (3.26) and (3.27) imply

$$\|F_n(t)\| = \|f_{n,0}\| + \int_0^t \left\|Q^+(s, F_n(s))\right\| \mathrm{d}s - \int_0^t \left\|Q^-(s, F_n(s))\right\| \mathrm{d}s \quad (t \geq 0), \tag{3.81}$$

To prove the theorem, we need show that $\{F_n(t)\}_n$ and $\{Q^\pm(t, F_n(t))\}_n$ are convergent, and, then we need to interchange the limits conveniently in (3.78) and (3.81).

To this end, first observe that since $\{f_{n,0}\}_n$ is positive and increasing, and each F_n is the limit of a sequence of the form (3.60), we obtain by a simple induction (which uses the positivity and isotonicity of B in (3.60)) that $\{F_n(t)\}_n$ is increasing. Thus, by (A_0), the positive sequence $\{\Lambda F_n(t)\}_n$ is also increasing. Then (A_0) and (3.80) give $\|F_n(t)\| \leq {\lambda_0}^{-1} \|\Lambda F_n(t)\| \leq {\lambda_0}^{-1} \|\Lambda f_{n,0}\| \leq {\lambda_0}^{-1} \|\Lambda f_0\|$. Hence, for each $t \geq 0$, both $\{F_n(t)\}_n$ and $\{\Lambda F_n(t)\}_n$ are convergent, because X is monotone complete. Moreover, as Λ is closed, the limit $f(t)$ of $\{F_n(t)\}_n$ satisfies $f(t) \in \mathcal{D}(\Lambda) \cap X_+$, and we have $\Lambda F_n(t) \nearrow \Lambda f(t)$ as $n \to \infty$. Then, also $\{Q^\pm(t, F_n(t))\}_n$ are increasing, and $Q^\pm(t, F_n(t)) \leq Q^\pm(t, f(t))$ a.e. In particular, $\|Q^\pm(t, F_n(t))\| \leq \|Q^\pm(t, f(t))\|$ a.e. Consequently, $Q^\pm(t, F_n(t)) \nearrow Q^\pm(t, f(t))$ as $n \to \infty$, t -a.e., because X is monotone complete and $Q^\pm(t, \cdot)$ are o-closed t-a.e.

Now, applying (A_2) and (3.80) we get

$$\left\|Q^-(t, f(t))\right\| = \lim_{n\to\infty} \left\|Q^-(t, F_n(t))\right\| \leq a(\|\Lambda f_0\|) \|\Lambda f_0\| \tag{3.82}$$

a.e., hence $Q^-(\cdot,f) \in L^1_{loc}(\mathbb{R}_+;X_+)$.

Thus we can take the limit $n \to \infty$ in (3.78) and (3.81), and we can apply, say, Lebesgue's theorem to the second term of (3.78) and (3.81), respectively. We obtain

$$f(t) = f_0 + \lim_{n\to\infty} \int_0^t Q^+(s,F_n(s))\mathrm{d}s - \int_0^t Q^-(s,f(s))\mathrm{d}s, \tag{3.83}$$

and, by (3.26),

$$\|f(t)\| = \|f_0\| + \lim_{n\to\infty} \int_0^t \left\|Q^+(s,F_n(s))\right\| \mathrm{d}s - \int_0^t \left\|Q^-(s,f(s))\right\| \mathrm{d}s. \tag{3.84}$$

Since $\|f(t)\| < \infty$ for $t \geq 0$, and $Q^-(\cdot,f) \in L^1_{loc}(\mathbb{R}_+;X_+)$, by (3.84), for each $t \geq 0$,

$$\lim_{n\to\infty} \int_0^t \left\|Q^+(s,F_n(s))\right\| \mathrm{d}s < \infty. \tag{3.85}$$

Hence, applying, e.g., the monotone convergence theorem, it follows that $Q^+(\cdot,f)$ is Bochner integrable and we can finally pass to the limit under the integral sign in (3.83), (3.84), (3.80), and in (3.79), to conclude the proof of theorem. □

Proof of Proposition 3.3

Equality (3.51) follows observing that $\Delta(s,F_n(s)) \equiv 0$ in (3.79), and taking the ∞ limit. As in the uniqueness part of the proof of Theorem 3.1, the solution f of (3.31) provided by Theorem 3.2 also solves the mild problem (3.57) (but here, $\Delta(t,f) = 0$ in the expression (3.55) of B, by virtue of (3.51)). Now the uniqueness follows by an argument similar to the one used in the uniqueness part of the proof of Theorem 3.1, taking now advantage of the property $\Delta(s,F_n(s)) \equiv 0$ (hence of (3.51)). □

Proof of Proposition 3.4

a) Let f_0, $\{f_{0,i}\}$, $\{f_{n,i}(t)\}_n$, and $\{F_i(t)\}_i$ be as in Step B of the proof of Theorem 3.1a). Then for each i, the sequence $\{\Gamma f_{n,i}(t)\}_n$ is positive and increasing. Moreover, it is norm-bounded because

$$\|\Gamma f_{n,i}(t)\| \leq \|\Gamma f_0\| \quad (t \geq 0), \tag{3.86}$$

as a consequence of (3.65) and of the property $\Gamma f_{0,i} \leq \Gamma f_0$.

As X is monotone complete, it follows that $\{\Gamma f_{n,i}(t)\}_n$ is convergent for all i.

Recall that Γ is closed, and $f_{n,i}(t) \nearrow F_i(t)$ as $n \to \infty$, for all i. Consequently, $F_i(t) \in \mathcal{D}(\Gamma)$ and $\Gamma f_{n,i}(t) \nearrow \Gamma F_i(t)$ as $n \to \infty$, $i = 1, 2, \ldots$. In addition, $\|\Gamma F_i\| \le \|\Gamma f_0\|$, $t \ge 0$,$i = 1, 2, \ldots$. Then, reasoning as before, we conclude that $f(t) \in \mathcal{D}(\Gamma)$, $\Gamma F_i(t) \nearrow \Gamma f(t)$ as $i \to \infty$, and that $\|\Gamma f\|$ satisfies (3.52).

b) The proof of (3.53) follows as in a), with the only remark that instead of (3.86), we make use of the inequalities

$$\|\Gamma f_{n,i}(t)\| \le \exp(\rho_\Gamma(\|\Lambda_1 f_{0,i}\|)t)\ \|\Gamma f_{0,i}\| \le \exp(\rho_\Gamma(\|\Lambda_1 f_0\|)t)\ \|\Gamma f_0\| \quad (t \ge 0), \tag{3.87}$$

which are immediate by (3.67), because ρ_Γ is non-decreasing. □

4. Applications

4.1. Smoluchowski's coagulation equation

For $k \ge 0$, let $L^1_k := L^1_k(\mathbb{R}_+; \mathrm{d}y)$ be the space of real measurable functions $g : \mathbb{R}_+ \mapsto \mathbb{R}$ such that

$$\|g\|_{L^1_k} := \int_{\mathbb{R}_+} (1+y)^k\, |g(y)|\, \mathrm{d}y < \infty. \tag{4.1}$$

Denote $L^1_{k,+} = \{g \in L^1_k : g \ge 0\}$. Consider problem (2.2) in the space $X = L^1(\mathbb{R}_+; \mathrm{d}y)$ (equipped with the usual norm $\|\cdot\| = \|\cdot\|_{L^1}$, and with the natural order $\le$).

Consider L^1_k as a subset of X. Let $i = 0, 1$ and define the positive linear operators $\Lambda_{c,i} : \mathcal{D}(\Lambda_{c,i}) \subset X \mapsto X$ by $\mathcal{D}(\Lambda_{c,i}) = L^1_{\gamma_i}$, $(\Lambda_{c,i} g)(y) := \lambda_i(y) g(y)$, with $\lambda_i(y) := (1+y)^{\gamma_i}$, $y \ge 0$ a.e., where $\gamma_0 = \beta$ and $\gamma_1 = \alpha + \beta$.

Note that (2.3) and (2.4) define Q_c^+ and Q_c^- as positive and isotone nonlinear operators in X, respectively, with the common domain $\mathcal{D}_c := L^1_\beta$.

Then the i.v.p. for (2.2) can be formulated in X as

$$\frac{\mathrm{d}}{\mathrm{d}t} f = Q_c(f) = Q_c^+(f) - Q_c^-(f) \quad f(0) = f_0, \quad t > 0. \tag{4.2}$$

In this case, one can apply Theorem 3.1a). The only point is to check that $\Lambda_{c,i}$ $(i = 0, 1)$ and $Q_c^\pm$ verify inequalities of the form (3.40) and (3.42). Indeed, if $g \in L^1_{2\beta,+}$, then starting from (2.7), we find

$$0 \le \left\|\Lambda_{c,i} Q_c^-(g)\right\| - \left\|\Lambda_{c,i} Q_c^+(g)\right\| =$$

$$= \frac{1}{2}\int_{\mathbb{R}_+^2} [(1+y)^{\gamma_i} + (1+y_*)^{\gamma_i} - (1+y+y_*)^{\gamma_i}] q(y,y_*) g(y) g(y_*) \mathrm{d}y \mathrm{d}y_*, \tag{4.3}$$

because $0 \le \gamma_i \le 1$, and

$$\frac{(1+y)^{\gamma} + (1+y_*)^{\gamma}}{(1+y+y_*)^{\gamma}} \ge \inf_{x \ge 0} \frac{1+x^{\gamma}}{(1+x)^{\gamma}} = 1 \quad (0 \le \gamma \le 1, \quad y, y' \ge 0). \tag{4.4}$$

Inequality (4.3) shows that $g \mapsto \Delta_c(g) := \|\Lambda_{c,0} Q_c^-(g)\| - \|\Lambda_{c,0} Q_c^+(g)\|$ defines a positive isotone map $\Delta_c : \mathcal{D}(\Delta_c) \mapsto \mathbb{R}$ with domain $\mathcal{D}(\Delta_c) = L^1_{2\beta,+}$.

Starting again from (2.7), we find that if $g \in L^1_{3\beta,+}$, then

$$\left\|\Lambda_{c,0}^2 Q_c^+(g)\right\| - \left\|\Lambda_{c,0}^2 Q_c^-(g)\right\| =$$

$$= \frac{1}{2}\int_{\mathbb{R}_+^2} \left[(1+y+y_*)^{2\beta} - (1+y)^{2\beta} - (1+y_*)^{2\beta}\right] q(y,y_*) g(y) g(y_*) \mathrm{d}y \mathrm{d}y_*. \tag{4.5}$$

If $0 \le \beta \le 1/2$, applying again (4.4) in (4.5), we get

$$\left\|\Lambda_{c,0}^2 Q_c^+(g)\right\| - \left\|\Lambda_{c,0}^2 Q_c^-(g)\right\| \le 0, \tag{4.6}$$

which is of the form (3.42) with $\rho \equiv 0$.

If $1/2 < \beta \le 1$, then to estimate (4.5), we apply the following form ([11]) of Povzner's algebraic inequality, which can be easily proved[8]:

$$(1+y+y_*)^{2\beta} - (1+y)^{2\beta} - (1+y_*)^{2\beta} \le 2(1+y)^{\beta}(1+y_*)^{\beta} \quad (y, y_* \ge 0). \tag{4.7}$$

Thus, applying (4.7) in (4.5), we find that there is a number $c > 0$ such that

$$\left\|\Lambda_{c,0}^2 Q_c^+(g)\right\| - \left\|\Lambda_{c,0}^2 Q_c^-(g)\right\| \le c \left\|\Lambda_{c,1} g\right\| \left\|\Lambda_{c,0}^2 g\right\|. \tag{4.8}$$

Clearly, inequality (4.8) is of the form (3.42) with $\rho(x) = cx$.

Let $a_c(x) := a_0 x$, for some constant $a_0 > 0$. If a_0 is sufficiently large, then the map $L^1_{\beta,+} \ni g \mapsto a_0 \|\Lambda_{c,0} g\| \Lambda_{c,0} g - Q_c^-(g) \in X$ has the properties required in (A_2).

It appears that $Q_c^{\pm}$, $\Lambda_{c,0}$, $\Lambda_{c,1}$ and a_c verify the conditions of Theorem 3.1a) for $Q^{\pm}$, Λ, Λ_1 and a, respectively, provided that a_0 is sufficiently large. Consequently, one can apply Theorem 3.1a) to the i.v.p. (4.2). We obtain

[8] Indeed, (4.7) is equivalent to $\zeta(x) = 2x^{\beta} + 1 + x^{2\beta} - (1+x)^{2\beta} \ge 0$ for all $x > 0$. However, as $\zeta(x^{-1}) = x^{-2\beta}\zeta(x)$, to prove that $\zeta(x) \ge 0$ for $x > 0$, we need only show that $\zeta(x) \ge 0$ on $(0,1]$, which is immediate, because $1/2 < \beta \le 1$.

THEOREM **4.1** *Let $f_0 \in L^1_{2\beta,+}$ in problem (4.2). Then Eq. (4.2) has a unique strong solution f such that $f(t) \in L^1_{2\beta,+}$, $t \geq 0$, and $\|f(t)\|_{L^1_{2\beta}}$ is locally bounded on $\mathbb{R}_+$. In addition $f, (1+y)^\beta f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+, \mathrm{d}y))$,*

$$\|f(t)\|_{L^1_\beta} + \int_0^t \Delta_c(f(s))\mathrm{d}s = \|f_0\|_{L^1_\beta} \quad (t \geq 0), \tag{4.9}$$

and there is a constant $c > 0$ such that

$$\|f(t)\|_{L^1_{2\beta}} \leq \exp(c\,\|f_0\|_{L^1_{\alpha+\beta}}\, t)\,\|f_0\|_{L^1_{2\beta}} \quad (t \geq 0). \tag{4.10}$$

Note here that if $0 \leq 2\beta < 1$, then Theorem 4.1 allows for the existence of solutions with infinite initial mass (see also [22]) i.e., $f_0 \in L^1_{2\beta,+}$, but $f_0 \notin L^1_1$. The theorem does not imply directly the mass conservation, except for the case $q_1 > 0$, $\beta = 1$ and $\alpha = 0$. However, if $f_0 \in L^1_{2\beta,+} \cap L^1_1$, then the solution $f(t)$ has finite mass: indeed, if $\Gamma : L^1_1 \subset L^1 \mapsto L^1$ is defined by $(\Gamma g)(y) = yg(y)$ a.e. on $\mathbb{R}_+$, then clearly, Γ is of type D on $\cap_{k=1}^\infty L^1_{k\beta,+}$, hence Prop. 3.4a) applies, so that $f \in L^1_{2\beta,+} \cap L^1_1$, and $\|\Gamma f(t)\| \leq \|\Gamma f_0\|$.

Theorem 4.1 remains valid in the case of the discrete Smoluchowski equation (2.10), with obvious change in formulation[9].

4.2. Povzner-like model with dissipative collisions

Let $X = L^1(\mathbb{R}^3 \times \mathbb{R}^3; \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}) = L^1$, equipped with the norm $\|\cdot\| := \|\cdot\|_{L^1}$ and the natural order $\leq$. Denote by $L^1_k := L^1_k(\mathbb{R}^3 \times \mathbb{R}^3; \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v})$, $k \in \mathbb{R}$, the space of measurable functions on $g : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}$ satisfying

$$\|g\|_{L^1_k} := \int_{\mathbb{R}_+} (1 + |\mathbf{v}|^2)^{\frac{k}{2}}\, |g(\mathbf{x}, \mathbf{v})|\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} < \infty. \tag{4.11}$$

As before, $L^1_{k,+}$ denotes the positive cone in L^1_k. It can be seen that (2.15) and (2.16) define $Q^\pm_d$ as positive and isotone operators on the common domain $\mathcal{D} := L^1_\gamma$. This follows easily if we perform the change of variable $(0, R] \times \Omega \ni (r, \mathbf{n}) \mapsto \mathbf{y} := r\mathbf{n} \in \{\mathbf{z} \in \mathbb{R}^3 : |\mathbf{z}| \leq R\}$ in (2.15) and (2.16), and then take into account (2.17).

Now, formulated in X, the i.v.p. (2.14) reads

$$\frac{\mathrm{d}}{\mathrm{d}t} f = Af + Q^+_d(f) - Q^-_d(f), \quad f(0) = f_0 \geq 0, \tag{4.12}$$

[9]Note that L^1_r, defined before, must be now replaced by $l^1_r(\mathbb{R}) = \{c = (c_j) : c_j \in \mathbb{R}, j = 1, 2, ..., \|c\|_r := \sum_{j=1}^\infty j^r |c_j| < \infty\}$, $r \geq 0$.

where $f = f(t, \mathbf{x}, \mathbf{v})$ is the one-particle distribution function, A is the infinitesimal generator of the C_0 group $(U^t f)(\mathbf{x}, \mathbf{v}) := f(\mathbf{x} - t\mathbf{v}, \mathbf{v})$, a.e.

Let the positive linear operator $\Lambda_d : L^1_2 \mapsto X$ be defined by $(\Lambda_d g)(\mathbf{x}, \mathbf{v}) := \lambda(\mathbf{v}) g(\mathbf{x}, \mathbf{v})$ a.e. on $\mathbb{R}^3 \times \mathbb{R}^3$, with $\lambda(\mathbf{v}) := (1 + |\mathbf{v}|^2)$. Define $a_d(x) := c_0 x$ for some constant $c_0 > 0$. If c_0 is sufficiently large, then a_d, Λ_d and $Q^{\pm}_d$ verify the conditions of Corollary 3.1 for a, $\Lambda = \Lambda_1$ and $Q^{\pm}$, respectively.

Indeed, the operators $Q^{\pm}_d$ are p-saturated. Moreover, they are o-closed, by the monotone convergence theorem. It is immediate that the domain conditions imposed in Corollary 3.1 are satisfied. Further, applying (2.12) in (2.18), we obtain an inequality of the form (3.40), i.e., if $g \in L^1_{4,+}$, then

$$0 \le \Delta_d(g) := \left\|\Lambda_d Q^-_d(g)\right\| - \left\|\Lambda_d Q^+_d(g)\right\| =$$

$$= \int_0^R \mathrm{d}r \int_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \pi(r, \mathbf{n}, \mathbf{v}, \mathbf{w}, \mathbf{x}) g(\mathbf{x}, \mathbf{v}) g(\mathbf{x} + r\mathbf{n}, \mathbf{w}) \mathrm{d}\mathbf{n} \mathrm{d}\mathbf{v} \mathrm{d}\mathbf{w} \mathrm{d}\mathbf{x}, \tag{4.13}$$

where $\pi(r, \mathbf{n}, \mathbf{v}, \mathbf{w}, \mathbf{x}) := \beta(\mathbf{n})(1 - \beta(\mathbf{n})) \left|\langle \mathbf{n}, \mathbf{v} - \mathbf{w}\rangle\right|^{2+\gamma} P(r, \mathbf{n})$. Remark here that the map $L^1_{4,+} \ni g \mapsto \Delta_d(g) \in \mathbb{R}$ is positive and isotone. Moreover, for c_0 sufficiently large, the map $L^1_{2,+} \ni g \mapsto c_0 \left\|\Lambda_d g\right\| \Lambda_d g - Q^-_d(g) \in X$ is also positive and isotone. Further, to obtain an inequality of the form (3.42), note that (2.12) gives $\lambda(\mathbf{v}')^2 + \lambda(\mathbf{w}')^2 \le (2 + |\mathbf{v}'|^2 + |\ \mathbf{w}'|^2)^2 \le (2 + |\mathbf{v}|^2 + |\mathbf{w}|^2)^2 = \lambda(\mathbf{v})^2 + \lambda(\mathbf{w})^2 + 2\lambda(\mathbf{v})\lambda(\mathbf{w})$, which can be applied in (2.18) to conclude easily that there are two constants c_1, $c > 0$ such that

$$\left\|\Lambda^2_d Q^+_c(g)\right\| - \left\|\Lambda^2_d Q^-_d(g)\right\| \le$$

$$\le c_1 \int_0^R \mathrm{d}r \int_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} r^2 \lambda(\mathbf{v}) \lambda(\mathbf{w})^{1+\frac{\gamma}{2}} g(\mathbf{x}, \mathbf{v}) g(\mathbf{x} + r\mathbf{n}, \mathbf{w}) \mathrm{d}\mathbf{n} \mathrm{d}\mathbf{v} \mathrm{d}\mathbf{w} \mathrm{d}\mathbf{x} \le$$

$$\le c \left\|\Lambda_d g\right\| \left\|\Lambda^2_d g\right\|, \tag{4.14}$$

for all $g \in L^1_{6,+}$. Finally, it is obvious that the group U^t (generated by A) commutes with the semigroup V^t generated by Λ_d, and $\Lambda^k Q^+(U^{\cdot} g) \in L^1_{loc}(\mathbb{R}_+; X_+)$ for all $g \in \cap_{n=1}^{\infty} L^1_{n,+}$, $k = 1, 2, \ldots..$

Therefore, by Corollary 3.1, we have the following result ([11]):

Theorem **4.2** *Let $f_0 \in L^1_{4,+}$ in problem* (4.12). *Then Eq.* (4.12) *has a unique positive mild solution f such that $f(t) \in L^1_{4,+}$, $t \ge 0$, and $\|f(t)\|_{L^1_4}$ is locally bounded on $\mathbb{R}_+$. In addition, f, $(1 + |\mathbf{v}|^2) f \in C(\mathbb{R}_+; L^1)$,*

$$\|f(t)\|_{L^1_2} + \int_0^t \Delta_d(f(s)) \mathrm{d}s = \|f_0\|_{L^1_2} \quad (t \ge 0), \tag{4.15}$$

and there is a constant $c > 0$ *such that*

$$\|f(t)\|_{L^1_4} \leq \exp(c\,\|f_0\|_{L^1_2}\,t)\,\|f_0\|_{L^1_4} \quad (t \geq 0). \tag{4.16}$$

The argument of Theorem 4.2 can be repeated with obvious modifications to provide a similar result for the space-homogeneous version of Eq. (2.14), which coincides with the force-free, three dimensional space-homogeneous Boltzmann model for granular flows, [5, 6].

4.3. Povzner-like model with chemical reactions

Let $X := L^1(\mathbb{R}^3 \times \mathbb{R}^3; d\mathbf{x}d\mathbf{v})^N$ be equipped with the order $\leq$ induced by the order of the components (i.e., the natural order of L^1). The norm on X is defined as

$$\|g\| := \sum_{i=1}^{N} \int_{\mathbb{R}^3\times\mathbb{R}^3} |g_i(\mathbf{x},\mathbf{v})|\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} = \sum_{i=1}^{N} \|g_i\|_{L^1}\,. \tag{4.17}$$

Denote by $L^1_k := L^1_k(\mathbb{R}^3\times\mathbb{R}^3; \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v})$, $k \in \mathbb{R}$, the space of measurable functions $g : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}$ satisfying

$$\|g\|_{L^1_k} := \int_{\mathbb{R}^3\times\mathbb{R}^3} (1 + |\mathbf{v}|^2)^{\frac{k}{2}}\,|g(\mathbf{x},\mathbf{v})|\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \tag{4.18}$$

and let $L^1_{k,+}$ be the positive cone in L^1_k.

It is natural to formulate the i.v.p. (2.29) in the space X.

Under the conditions of the model, (2.30) and (2.31) define Q_i^+ and Q_i^-, $1 \leq i \leq N$, as operators from the common domain $(L^1_2)^N \subset X$ to $L^1(\mathbb{R}^3; \mathrm{d}\mathbf{v})$. Defining the operators $Q_B^\pm : (L^1_2)^N \subset X \mapsto X$ by $Q_B^\pm = (Q_1^\pm, \ldots.., Q_N^\pm)$, we can write the i.v.p. for Eq. (2.29) in X as

$$\frac{\mathrm{d}}{\mathrm{d}t} f + A = Q_B^+(t,f) - Q_B^-(t,f), \quad 0 \leq f(0) = f_0 \in X \quad (t > 0), \tag{4.19}$$

where A is the infinitesimal generator of the C_0 group of isometries $\{U^t\}_{t\in\mathbb{R}}$ on X, given by $(U^t f)(\mathbf{x},\mathbf{v}) := f((\mathbf{x} - t\mathbf{v}, \mathbf{v})$.

Define the positive closed linear operator $\Lambda_B : (L^1_2)^N \mapsto X$ by $(\Lambda_B g)_i(\mathbf{v}) = \lambda_i(\mathbf{v})g(\mathbf{v})$ a.e. on $\mathbb{R}^3 \times \mathbb{R}^3$, where $\lambda_i(\mathbf{v}) := m_i + m_i\,|\mathbf{v}|^2/2 + E_i$, $1 \leq i \leq N$. One can state the following result ([12]):

THEOREM **4.3** *Suppose that in problem* (4.19), $f_{0,i} \in L^1_{4,+}$, $1 \le i \le N$. *Then Eq.* (4.19) *has a unique mild solution* $f(t) = (f_1, ..., f_N)$ *such that* $f_i(t) \in L^1_{4,+}$, $t \ge 0$, *and* $\|f_i(t)\|_{L^1_4}$ *is locally bounded on* $\mathbb{R}_+$, $1 \le i \le N$. *In addition,* f_i, $(1+|\mathbf{v}|^2)f_i \in C(\mathbb{R}_+; L^1)$, $1 \le i \le N$,

$$\|\Lambda_B f(t)\| = \|\Lambda_B f_0\| \quad (t \ge 0), \tag{4.20}$$

and there is a constant $\rho_0 > 0$ *such that*

$$\left\|\Lambda_B^2 f(t)\right\| \le \exp(\rho_0 \|\Lambda_B f_0\| \, t) \left\|\Lambda_B^2 f_0\right\| \quad (t \ge 0). \tag{4.21}$$

The above result follows by applying Theorem 3.1 in the case $\Lambda = \Lambda_1 = \Lambda_B$. Indeed, the domain conditions of Theorem 3.1, as well as properties (A_0), (A_1) can be immediately checked (with $\Delta = 0$, owing to (2.38). Next, let $a_0 > 0$ be some constant, and define $a(x) := a_0 x$. Owing to (2.38), for a_0 sufficiently large, the map $L^1_{2,+} \ni g \to a_0 \|\Lambda_B g\| \Lambda_B g - Q^-(g) \in X$ satisfies (A_2). Finally, note that, as a consequence of (2.39) (and of (2.37)), there exists a number $\rho_0 > 0$ such that

$$\sum_{i=1}^N \int_{\mathbb{R}^3} (\Psi_i^{(0)} + \Psi_i^{(4)})^2 \left[Q_i^+(g) - Q_i^-(g)\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \le$$

$$\le \rho_0 \left\|(1+|\mathbf{v}|^4)g\right\|_{L^1} \left\|(1+|\mathbf{v}|^2)g\right\|_{L^1}, \tag{4.22}$$

for, say, all $g \in (L^1_{6+})^N$.

Then inequality (3.13) gives exactly (A_3) with $\rho(x) := \rho_0 x$.

4.4. Boltzmann model with inelastic collisions and reactions

Let $X := (L^1(\mathbb{R}^3; \mathrm{d}\mathbf{v}))^N$ be equipped with the order $\le$ induced by the order of the components (i.e., the natural order of L^1). The norm on X is defined as

$$\|g\| := \sum_{i=1}^N \int_{\mathbb{R}^3} |g_i(\mathbf{v})| \, \mathrm{d}\mathbf{v} = \sum_{i=1}^N \|g_i\|_{L^1}. \tag{4.23}$$

Denote by $L^1_k := L^1_k(\mathbb{R}^3; \mathrm{d}\mathbf{v})$, $k \in \mathbb{R}$, the space of measurable functions $g : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}$ satisfying

$$\|g\|_{L^1_k} := \int_{\mathbb{R}_+} (1+|\mathbf{v}|^2)^{\frac{k}{2}} |g(\mathbf{v})| \, \mathrm{d}\mathbf{v} < \infty \tag{4.24}$$

and let $L^1_{k,+}$ be the positive cone in L^1_k.

It is natural to formulate the i.v.p. for Eq. (2.47) in the space X. Under the above conditions, (2.48) and (2.49) define Q_i^+ and Q_i^-, $1 \leq i \leq N$, respectively, as operators from the common domain $\mathcal{D} = (L^1_2)^N \subset X$ to $L^1(\mathbb{R}^3; d\mathbf{v})$. Defining $Q_B^\pm : \mathcal{D} \subset X \mapsto X$ by $Q_B^\pm = (Q_1^\pm, ..., Q_N^\pm)$, we can write the i.v.p. for Eq. (2.47) in X

$$\frac{\mathrm{d}}{\mathrm{d}t} f = Q_B^+(f) - Q_B^-(f), \quad f(0) = f_0 = (f_{0,1}, ..., f_{0,N}) \in X_+. \tag{4.25}$$

We shall prove the existence of solutions to problem (4.25), by applying Theorem 3.1a) (in the case $\Lambda = \Lambda_1$). To this end, let the positive closed linear operator $\Lambda_B : (L^1_2)^N \mapsto X$ be defined on components by $(\Lambda_B g)_i(\mathbf{v}) = \lambda_i(\mathbf{v}) g(\mathbf{v})$ a.e. on $\mathbb{R}^3 \times \mathbb{R}^3$, where $\lambda_i(\mathbf{v}) := m_i + m_i |\mathbf{v}|^2 /2 + E_i$, $1 \leq i \leq N$. Denote $l_\gamma(\mathbf{w}) := \sum_{i \in \mathcal{N}(\gamma)} \sum_{j=1}^{\gamma_i} \lambda_i(\mathbf{w}_{i,j})$; $\gamma \in \mathcal{M}$. Then clearly, $l_\gamma(\mathbf{w}) = M_\gamma + W_\gamma(\mathbf{w})$, hence

$$0 \leq W_\gamma(\mathbf{w}) < l_\gamma(\mathbf{w}). \tag{4.26}$$

In addition, defining $\lambda^\gamma(\mathbf{w}) := \prod_{i \in \mathcal{N}(\gamma)} \prod_{j=1}^{\gamma_i} \lambda_i(\mathbf{w}_{i,j})$, $\gamma \in \mathcal{M}$, we have

$$l_\gamma(\mathbf{w}) \leq |\gamma| \, E^{1-|\gamma|} \lambda^\gamma(\mathbf{w}), \tag{4.27}$$

where $E := \min\{m_i + E_i : 1 \leq i \leq N\}$. It is useful to remark that, since $W_\gamma(\mathbf{w}) \geq E \, |\gamma| > 0$, and $0 \leq q \leq 1$, then by (2.56), (4.26) and (4.27),

$$\nu_{\beta,\alpha}(\mathbf{w}) \leq C \lambda^\alpha(\mathbf{w}) \quad (\mathbf{w} \in \mathbb{R}^{|\alpha|}, \, a.e.), \tag{4.28}$$

for all $\alpha, \beta \in \mathcal{M}$. Here $C = C(E, K) > 0$ is a number depending on E and K (recall that K is the maximum number of partners in a reaction channel).

To apply Theorem 3.1a) to (4.25), first remark that $Q_B^\pm$ and Λ_B verify the domain conditions imposed to $Q^\pm$ and Λ by the theorem. Moreover, Λ_B has the properties required for Λ in (A_0). Further, observe that formula (2.57) provides a correspondent to (3.40), specifically,

$$\Delta_B(g) := \left\| \Lambda_B Q_B^-(g) \right\| - \left\| \Lambda_B Q_B^+(g) \right\| = 0 \quad (g \in (L^1_{4,+})^N). \tag{4.29}$$

To obtain a correspondent to (3.42), let $s_\gamma(\mathbf{w}) := \sum_{i \in \mathcal{N}(\gamma)} \sum_{j=1}^{\gamma_i} \lambda_i(\mathbf{w}_{i,j})^2$. Next, using the definition of Q_B^+ and property (B_2), and applying the obvious inequality $s_\alpha(\mathbf{w}) \leq l_\alpha(\mathbf{w})^2$, we find that if $g \in (L^1_{6,+})^N$, then

$$\left\| \Lambda_B^2 Q_B^+(g) \right\| = \sum_{\alpha,\beta \in \mathcal{M}} \int_{\mathbb{R}^{3|\alpha|} \times \Omega_\beta} s_\alpha(\mathbf{w}) p_{\beta,\alpha}(\mathbf{w}, \mathbf{n}) (g^\beta \circ u_{\beta,\alpha})(\mathbf{w}, \mathbf{n}) \mathrm{d}\mathbf{w} \mathrm{d}\mathbf{n} \leq$$

$$\leq \sum_{\alpha,\beta\in\mathcal{M}} \int_{\mathbb{R}^{3|\alpha|}\times\Omega_\beta} l_\alpha(\mathbf{w})^2 p_{\beta,\alpha}(\mathbf{w},\mathbf{n})(g^\beta \circ u_{\beta,\alpha})(\mathbf{w},\mathbf{n})\mathrm{d}\mathbf{w}\mathrm{d}\mathbf{n}. \tag{4.30}$$

We apply property (3.9) in the last integral. Then interchanging α and β, we get

$$\left\|\Lambda_B^2 Q_B^+(g)\right\| \leq \sum_{\alpha,\beta\in\mathcal{M}} \int_{\mathbb{R}^{3|\alpha|}\times\Omega_\beta} (l_\beta \circ u_{\beta,\alpha})^2(\mathbf{w},\mathbf{n}) r_{\beta,\alpha}(\mathbf{w},\mathbf{n}) g^\alpha(\mathbf{w})\mathrm{d}\mathbf{w}\mathrm{d}\mathbf{n}. \tag{4.31}$$

Since $l_\beta(\mathbf{w}) = M_\beta + W_\beta(\mathbf{w})$, property (B_3) implies that $(l_\beta \circ u_{\beta,\alpha})(\mathbf{w},\mathbf{n}) = l_\alpha(\mathbf{w})$ for all $(\alpha,\beta) \in \mathcal{C}_M$, $\mathbf{w} \in D^+_{\beta,\alpha}$. This and (B_1) enable us to deduce from (4.31) that

$$\left\|\Lambda_B^2 Q_B^+(g)\right\| \leq \sum_{\alpha,\beta\in\mathcal{M}} \int_{\mathbb{R}^{3|\alpha|}\times\Omega_\beta} l_\alpha(\mathbf{w})^2 r_{\beta,\alpha}(\mathbf{w},\mathbf{n}) g^\alpha(\mathbf{w})\mathrm{d}\mathbf{w}\mathrm{d}\mathbf{n}. \tag{4.32}$$

Now, using the definitions of $l_\alpha(\mathbf{w})$ and Q_B^-, and then, taking advantage of (2.56) and (4.26), we obtain from (4.32)

$$\left\|\Lambda_B^2 Q_B^+(g)\right\| \leq$$

$$\leq \sum_{\alpha,\beta\in\mathcal{M}} \int_{\mathbb{R}^{3|\alpha|}\times\Omega_\beta} s_\alpha(\mathbf{w}) r_{\beta,\alpha}(\mathbf{w},\mathbf{n}) g^\alpha(\mathbf{w})\mathrm{d}\mathbf{w}\mathrm{d}\mathbf{n} + \rho_B(\|\Lambda_B g\|)\left\|\Lambda_B^2 g\right\| =$$

$$= \left\|\Lambda_B^2 Q_B^-(g)\right\| + \rho_B(\|(\Lambda_B g\|)\left\|\Lambda_B^2 g\right\|, \tag{4.33}$$

where ρ_B is a positive non-decreasing (polynomial) function.

Therefore, the last inequality is the required correspondent to (3.42) (in the case $\Lambda = \Lambda_1$).

Further, let $a_0 > 0$ be some constant, and define $a(x) := a_0 \sum_{p=1}^{NK} x^p$, $x \geq 0$. Therefore, $a(\|\Lambda_B g\|) = a_0 \sum_{p=1}^{NK} \|\Lambda_B g\|^p$. But each term $\|\Lambda_B g\|^p$ in the r.h.s of the last equality can be expressed by (4.23), and the resulting expression can be expanded by the multinomial formula. Then, after some elementary algebra we get the following useful expression

$$a(\|\Lambda_B g\|) = a_0 \sum_{\gamma\in\mathcal{M},\,|\gamma|\geq 1} c_{\gamma,i} \int_{\mathbb{R}^{3|\gamma|}} \lambda^\gamma(\mathbf{w}) g^\gamma(\mathbf{w})\mathrm{d}\mathbf{w}, \tag{4.34}$$

where $c_{\gamma,i} > 0$ are strictly positive, constant coefficients, $\gamma \in \mathcal{M}$, $|\gamma| \geq 1$, $1 \leq i \leq N$.

We show that if a_0 is large enough, then $(L^1_{2,+})^N \ni g \mapsto a(\|\Lambda_B f\|)\Lambda_B g - Q^-_B(g) \in X$ is positive and isotone. To this end, first note that one can write

$$Q_i^-(g)(\mathbf{v}) = R_i(g)(\mathbf{v})\, g_i(\mathbf{v}), \quad (g \in (L^1_{2,+})^N,\ \mathbf{v} \in \mathbb{R}^3\ a.e.,\ 1 \le i \le N), \tag{4.35}$$

where

$$R_i(g)(\mathbf{v}) := \sum_{\alpha,\beta \in \mathcal{M}} \alpha_i \int\limits_{\mathbb{R}^{3|\alpha|-3}} \left[\nu_{\beta,\alpha}(\mathbf{w}) \prod_{\substack{s \in \mathcal{N}(\alpha) \\ (s,j) \neq (i,\alpha_i)}} \prod_{j=1}^{\alpha_s} g_s(\mathbf{w}_{s,j}) \right]_{\mathbf{w}_{i,\alpha_i} = \mathbf{v}} \mathrm{d}\tilde{\mathbf{w}}_i, \tag{4.36}$$

with $\nu_{\beta,\alpha}$ as in (2.56). Hence,

$$a(\|\Lambda_B g\|)(\Lambda_B g)_i(\mathbf{v}) - Q_i^-(g)(\mathbf{v}) = [a(\|\Lambda_B g\|)\lambda_i(\mathbf{v}) - R_i(g)(\mathbf{v})]\, g_i(\mathbf{v}). \tag{4.37}$$

It is convenient to set

$$R_i^A(g)(\mathbf{v}) := C \sum_{\alpha,\beta \in \mathcal{M}} \alpha_i \int\limits_{\mathbb{R}^{3|\alpha|-3}} \left[\lambda^\alpha(\mathbf{w}) \prod_{\substack{s \in \mathcal{N}(\alpha) \\ (s,j) \neq (i,\alpha_i)}} \prod_{j=1}^{\alpha_s} g_s(\mathbf{w}_{s,j}) \right]_{\mathbf{w}_{i,\alpha_i} = \mathbf{v}} \mathrm{d}\tilde{\mathbf{w}}_i, \tag{4.38}$$

with C as in (4.28). Summing on β in (4.38), using the explicit form of $\lambda^\alpha(\mathbf{w})$, and invoking property (B_1), we are easily led to

$$R_i^A(g)(\mathbf{v}) = C\lambda_i(\mathbf{v}) \sum_{\gamma \in \mathcal{M},\, |\gamma| \ge 1} q_{\gamma,i} \int\limits_{\mathbb{R}^{3|\gamma|}} \lambda^\gamma(\mathbf{w}) g^\gamma(\mathbf{w})\, \mathrm{d}\mathbf{w}, \tag{4.39}$$

where $q_{\gamma,i} \ge 0$ are constant coefficients, $\gamma \in \mathcal{M}$, $|\gamma| \ge 1$, $1 \le i \le N$.

We introduce (4.34) and (4.38) in (4.37). Consequently, for $\mathbf{v} \in \mathbb{R}^3$ a.e.,

$$a(\|\Lambda_B g\|)(\Lambda_B g)_i(\mathbf{v}) - Q_i^-(g)(\mathbf{v}) = [R_i^A(g)(\mathbf{v}) - R_i(g)(\mathbf{v})] g_i(\mathbf{v}) + T_i(g)(\mathbf{v}), \tag{4.40}$$

where

$$T_i(g)(\mathbf{v}) := \lambda_i(\mathbf{v}) g_i(\mathbf{v}) \sum_{\gamma \in \mathcal{M},\, |\gamma| \ge 1} (a_0 c_{\gamma,i} - C q_{\gamma,i}) \int\limits_{\mathbb{R}^{3|\gamma|}} \lambda^\gamma(\mathbf{w}) g^\gamma(\mathbf{w}) \mathrm{d}\mathbf{w}. \tag{4.41}$$

Now we compare (4.36) and (4.38), by taking advantage of (4.28). It follows that the map $(L^1_{2,+})^N \ni g \mapsto [R_i^A(g) - R_i(g)] g_i \in L^1$ is positive and isotone, $1 \le i \le N$. Moreover, because of the form of $T_i(g)$, if $a_0 > 0$ is sufficiently large, then the mapping $(L^1_{2,+})^N \ni g \mapsto T_i(g)(\mathbf{v}) \in L^1$ is positive and isotone for all i. In this case, by virtue of (4.40), the map $(L^1_{2,+})^N \ni g \mapsto a(\|\Lambda_B g\|)\Lambda_B g - Q^-_B(g) \in X$ is also positive and isotone.

In conclusion, the conditions of Theorem 3.1a) are fulfilled (in the case $\Lambda = \Lambda_1$), so that we are in position to state the following result ([11]):

Theorem **4.4** *Suppose that in problem* (4.25), $f_{0,i} \in L^1_{4,+}$, $1 \le i \le N$. *Then Eq.* (4.25) *has a unique strong solution* $f(t) = (f_1, ..., f_N)$ *such that* $f_i(t) \in L^1_{4,+}$, $t \ge 0$, *and* $\|f_i(t)\|_{L^1_4}$ *is locally bounded on* $\mathbb{R}_+$, $1 \le i \le N$. *In addition,* f_i, $(1 + |\mathbf{v}|^2)f_i \in C(\mathbb{R}_+; L^1)$, $1 \le i \le N$,

$$\|\Lambda_B f(t)\| = \|\Lambda_B f_0\| \quad (t \ge 0), \tag{4.42}$$

and there is a non-decreasing function $\rho_B : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *such that*

$$\left\|\Lambda^2_B f(t)\right\| \le \exp(\rho_B(\|f_0\|)t) \left\|\Lambda^2_B f_0\right\| \quad (t \ge 0). \tag{4.43}$$

Theorem 4.4 does not state the conservation of mass, momentum and energy, but the conservation (in arbitrary units) of the quantity mass+(total) energy. However, the properties of $f(t)$, cf. Theorem 4.4, allow for checking immediately the separate conservation for each of the above quantities.

Theorem 4.4 reduces to the main monotonicity result of [2] when Eq. (4.25) is particularized to the case of the classical Boltzmann equation. Moreover, in that case, using suitable additional Povzner-like estimations, we can re-obtain the general moment estimations of [2], as application of Prop. 3.4b).

Finally, remark that similar analyses as for Theorems 4.2 and 4.4 can be developed for the main model considered, e.g., in [27].

4.5. Nonlinear von Neumann-Boltzmann equation

As Λ is unbounded (by construction), the existence of solutions to problem (2.62) seems not immediate from general considerations.

However, one can show that the conditions of Theorem 3.1 are fulfilled with $a(x) = x$.

First recall that $\mathrm{Tr}[\Lambda^k(Q^+ - Q^-)](F) = 0$, for all $0 \le F \in \mathcal{D}(\Lambda^k) \cap X_+$, $k = 0, 1$. Then observe that, since $\Lambda \ge \mathbb{I}$, it follows easily that $\mathrm{Tr}[\Lambda^2(Q^+ - Q^-)](F) \le \varepsilon \mathrm{Tr}(\Lambda F)\mathrm{Tr}F \le \varepsilon \mathrm{Tr}(\Lambda F)\mathrm{Tr}(\Lambda^2 F)$ for all $0 \le F \in \mathcal{D}(\Lambda^3) \cap X_+$.

So we can now formulate our existence result ([12]):

Theorem **4.5** *Suppose that in problem* (2.62)*,* $0 \le F_0 \in \mathcal{D}(\Lambda^2)$*. Then Eq.* (2.62) *has a unique mild solution* $0 \le F(t) \in \mathcal{D}(\Lambda^2)$*, and* $\mathrm{Tr}F(t)$ *is locally bounded. Moreover,* $F, \Lambda F \in C(\mathbb{R}_+; X)$*,* $\mathrm{Tr}F(t) = \mathrm{Tr}F_0$*,* $\mathrm{Tr}(\Lambda F)(t) = \mathrm{Tr}(\Lambda F_0)$ *and* $\mathrm{Tr}(\Lambda^2 F)(t) \le \exp(t\varepsilon \mathrm{Tr}(\Lambda F_0))\mathrm{Tr}(\Lambda^2 F_0) \quad (t \ge 0)$.

5. Concluding remarks

The results of the previous section of applications can be easily completed taking advantage of Theorem 3.2. As an example, the previous Theorem 4.1 can be completed as follows

Proposition **5.1** *Let* $f_0 \in L^1_{\beta,+}$ *in problem* (4.2)*. Then Eq.* (4.2) *has a strong solution* $f(t) \in L^1_{\beta,+}$*,* $t \ge 0$*.*

As mentioned before, the uniqueness is no longer ensured in the latter case. Theorem 3.2 extends the main existence result of [11]. The other general existence results formulated in [11] can be similarly completed, with obvious modifications. This allows to reconsider the applications of [11], accordingly, in an obvious manner.

Prop. 3.3 provides uniqueness of the solutions in the special case when Δ vanishes on a rather large set. This can be applied, for instance, to the space-homogeneous Boltzmann equation with hard potentials, to obtain a similar existence result as in, e.g., [20]. However, in a more general case, the uniqueness problem, under the conditions of Theorem 3.2, remains open. Here we can however remark that the regularity conditions required in the theorem might be necessary to ensure the uniqueness of the *strong* solutions. Indeed, examples of non-unique (but) less regular solutions of the Boltzmann equation have been recently discovered, [26], [19].

In this chapter, we presented various examples of existence results for generalized Boltzmann models obtained by monotonicity methods. The above methods are potentially applicable to investigate other evolution problems.

On the other hand, the results presented in this review describe only partially the properties of the models considered. They must be completed by a thorough study of other properties of the models, e.g. the existence of stationary or/and equilibrium solutions, Lyapunov functionals, H-theorems (see e.g. [7]), asymptotic properties, construction of effective numerical methods.

6. Appendix

1) *Sketch of the Proof of Lemma* 3.3

Property $B(\cdot, g_i, h_j) \in L^1_{loc}(\mathbb{R}_+; X_+)$, $i, j = 1, 2$, follows from (A_1), (A_2) and Remark 3.2.

To prove (3.58), let

$$y_i(t) := \int_0^t \Delta(s, h_i(s)) \mathrm{d}s \quad (i = 1, 2). \tag{6.1}$$

Clearly, $0 \leq y_1(t) \leq y_2(t)$, because of the isotonicity of $\Delta(t, \cdot)$ (cf. (A_1)). Further, define $F(x, y) := a(x + y) - a(x)$, with a as in (A_2). The properties of a (cf.(A_2)) imply

$$F(x^*, y) - F(x, y) = \int_0^y \left[a'(x^* + \xi) - a'(x + \xi)\right] \mathrm{d}\xi \geq 0 \tag{6.2}$$

for all $0 \leq x \leq x^*$ and $y \geq 0$. Then one can show easily (invoking (A_2), the isotonicity of $Q^+(t, \cdot)$ and the obvious inequality $\Lambda g_1(t) \leq \Lambda g_2(t)$) that

$$\begin{aligned} 0 \leq B(t, g_1, h_1) = B(t, g_1, 0) + F\left(\|\Lambda g_1(t)\|, y_1(t)\right) \Lambda g_1(t) \leq \\ \leq B(t, g_2, 0) + F\left(\|\Lambda g_1(t)\|, y_1(t)\right) \Lambda g_2(t) \end{aligned} \tag{6.3}$$

and

$$0 \leq F\left(\|\Lambda g_1(t)\|, y_1(t)\right) \leq F\left(\|\Lambda g_2(t)\|, y_1(t)\right) \leq F\left(\|\Lambda g_2(t)\|, y_2(t)\right). \tag{6.4}$$

Inequalities (6.3) and (6.4) can be now easily combined to obtain (3.58). □

2) *Sketch of the Proof of Lemma* 3.4

a) Since $\mathcal{D}_+^\infty$ is p-saturated and $\Lambda^k Q^\pm(t, \cdot)$ are positive and isotone, the key point is to show that for each $T > 0$ and $n = 1, 2, ...$, there is $g_{n,T} \in \mathcal{D}_+^\infty$ such that

$$0 \leq f_n(t) \leq g_{n,T} \quad (0 \leq t \leq T \quad a.e.). \tag{6.5}$$

Then (3.41) gives $Q^-(t, g_{n,T}) \in \mathcal{D}_+^\infty$ a.e. on $\mathbb{R}_+$, hence $\Lambda^k Q^-(\cdot, g_{n,T}) \in L^1_{loc}(\mathbb{R}_+; X_+)$ for all $k = 0, 1, 2, ...$ The same properties hold for $Q^+(t, g_{n,T})$ and $\Lambda^k Q^+(\cdot, g_{n,T})$, respectively (by virtue of the assumptions of Theorem 3.1a) and by (3.44)).

Inequality (6.5) can be proved by induction.

Indeed, note that (6.5) is trivially verified for $n = 1$ by $g_{1,T} := 0$, and for $n = 2$ by $g_{2,T} := f_0$. Further, at the induction step, assuming that (6.5) is fulfilled for $n = 1, 2, ..q-1$ (with $q \geq 3$) applying, in essence, the properties of Δ, a, and (3.28), one first obtains

$$\Lambda^k \int_0^t B(s, g_{n-1,T}, g_{n-2,T})\mathrm{d}s = \int_0^t \Lambda^k B(s, g_{n-1,T}, g_{n-2,T})\mathrm{d}s \quad (0 \leq t \leq T), \tag{6.6}$$

for all $k = 1, 2, ...$ and $n = 1, 2, ..., q-1$. Then observe that $f_{q-1}(t) \leq g_{q-1,T}$ and $f_{q-2}(t) \leq g_{q-2,T}$ satisfy the conditions of Lemma 3.3 for $g_1 \leq g_2$ and $h_1 \leq h_2$, respectively. Thus, applying conveniently (3.56) and (3.58) in (3.60), and invoking (6.6), we get

$$0 \leq f_q(t) \leq f_0 + \int_0^T B(s, g_{q-1,T}, g_{q-2,T})\mathrm{d}s := g_{q,T} \in \mathcal{D}_+^\infty \quad (0 \leq t \leq T). \tag{6.7}$$

b) As before, it is sufficient to show by induction that property (6.5) is verified by $g_{n,T} \in \mathcal{D}(\Lambda^3) \cap X_+$.

First note that if $g_{1,T} = 0$ and $g_{2,T} = f_0$, then (6.5) is trivially verified for $n = 1, 2$, respectively.

The induction step is simpler than in a), because now one can make use of the fact that V^t is C_0. Then, $\int_0^t V^s h \mathrm{d}s \in \mathcal{D}(\Lambda)$ for all $h \in X$, $t \geq 0$, which, in our case, implies (for any $0 \leq t \leq T$)

$$\int_0^t V^{t-s} B(T, g_{q-1,T}, g_{q-2,T})\mathrm{d}s = \int_0^t V^s B(T, g_{q-1,T}, g_{q-2,T})\mathrm{d}s \in \mathcal{D}(\Lambda^3) \cap X_+. \tag{6.8}$$

Since, in our case, $B(t, g_{q-1,T}, g_{q-2,T}) \leq B(T, g_{q-1,T}, g_{q-2,T})$, we conclude the induction step, using property (6.8) with the key inequality

$$0 \leq f_q(t) \leq f_0 + \int_0^t V^{t-s} B(T, g_{q-1,T}, g_{q-2,T})\mathrm{d}s \quad (0 \leq t \leq T), \tag{6.9}$$

which follows, in essence, by Lemma 3.3, and by applying (3.56) and (3.58) in (3.60).

c) The statement follows from simple regularity considerations and some direct computation.

d) Obviously, $0 = f_1(t) \leq f_2(t) \leq f_3(t)$ a.e.. Then a straightforward induction, applying (3.58), shows that $\{f_n(t)\}$ is a.e. increasing.

For the rest of the proof, note that (3.63) implies (3.64). Inequality (3.63) can be proved by induction. Indeed, since $0 = f_1 \leq f_2(t) \leq f_0$, and $\Delta(t, 0) = 0$ a.e. (cf. Remark 3.1), formula (3.63) is trivially verified for $n = 2$. Let $q \geq 3$

and suppose inequality (3.63) to be valid for $n = 2, 3, ..., q-1$. If $n = q$ in (3.62), then the positivity of a and $0 \le \Lambda f_{q-1}(t) \le \Lambda f_q(t)$ give

$$f_q(t) \le f_0 + \int_0^t Q(s, f_{q-1}(s))\mathrm{d}s +$$

$$+ \int_0^t \left[a\left(\|\Lambda f_{q-1}(s)\| + \int_0^s \Delta(\tau, f_{q-2}(\tau))d\tau \right) - a\left(\|\Lambda f_0\|\right)\right] \Lambda f_q(s)\mathrm{d}s. \tag{6.10}$$

According to the induction hypothesis, (3.63) holds true for $n = q-1$. Hence (3.64) is also valid for $n = q-1$, as concluded before. Then $a(\|\Lambda f_{q-1}(s)\| + \int_0^s \Delta(\tau, f_{q-2}(\tau))d\tau)) \le a\left(\|\Lambda f_0\|\right)$, because a is non-decreasing. As $\Lambda f_q(s)$ is positive, clearly the integral term containing $\Lambda f_q(s)$, in the r.h.s. of (6.10) is negative. Then (3.63) becomes true for $n = q$.

e) Note that $Q^{\pm}(t, f_n(t)) \in \mathcal{D}(\Gamma)$, for a.e. $t \ge 0$. Also, $\Gamma Q^{\pm}(\cdot, f_n(\cdot)) \in L^1_{loc}(\mathbb{R}_+; X_+)$. Indeed, let $T > 0$ and $g_{n,T} \ge f_n(t)$ be as in a). If Γ is of type D on $\mathcal{D}^{\infty}_+$ (on $\mathcal{D}(\Lambda^2) \cap X_+$), then (3.36) and (3.41) give $\|\Gamma Q^{\pm}(t, f_n(t))\| \le \|\Gamma Q^{\pm}(t, g_{n,T})\| \le \|\Gamma Q^{-}(t, g_{n,T})\| \le a(\|g_{n,T}\|)\,\|\Gamma\Lambda g_{n,T}\|$ for a.e. $0 \le t \le T$. On the other hand, if Γ satisfies (3.46), then (3.41) implies

$$\left\|\Gamma Q^{+}(t, f_n(t))\right\| \le \left\|\Gamma Q^{-}(t, f_n(t))\right\| + \rho_\Gamma(\|\Lambda_1 g_{n,T}\|)\,\|\Gamma g_{n,T}\| \le$$

$$\le a(\|g_{n,T}\|)\,\|\Gamma\Lambda g_{n,T}\| + \rho_\Gamma(\|\Lambda_1 g_{n,T}\|)\,\|\Gamma g_{n,T}\| \quad (0 \le t \le T \quad a.e.).$$

But (3.63) is of the form (3.37), and the above considerations show that Lemma 3.2 applies (with Γ instead of Λ). Hence,

$$\|\Gamma f_n(t)\| + \int_0^t \Delta(s, f_{n-1}(s); \Gamma, Q)\mathrm{d}s \le \|\Gamma f_0\| \quad (t \ge 0, \quad n \ge 2). \tag{6.11}$$

Now the proof can be immediately concluded: if $n = 1$, then formula (3.65) is trivially satisfied; if $n \ge 2$, then (3.65) is directly implied by (6.11).

To obtain (3.66) observe that Λ^2 satisfies the conditions for Γ in e).

f) First apply inequality (3.46) in (6.11). It follows that

$$\|\Gamma f_n(t)\| \le \|\Gamma f_0\| + \int_0^t \rho_\Gamma(\|\Lambda_1 f_{n-1}(s)\|)\,\|\,\Gamma f_{n-1}(s)\|\,\mathrm{d}s \quad (t \ge 0, \quad n \ge 2). \tag{6.12}$$

But Λ_1 satisfies the conditions of e) in the present lemma, hence $\|\Lambda_1 f_n(t)\| \le \|\Lambda_1 f_0\|$, $t \ge 0$, $n = 1, 2, ...$. Introducing the last inequality in (4.16), we obtain

$$\|\Gamma f_n(t)\| \le \|\Gamma f_0\| + \rho_\Gamma(\|\Lambda_1 f_0\|) \int_0^t \|\Gamma f_{n-1}(s)\|\,\mathrm{d}s \quad (t \ge 0, \quad n \ge 2). \tag{6.13}$$

Finally, since (3.67) is obviously satisfied for $n = 1, 2$, a straightforward (Gronwall type) induction in (6.13) concludes the proof. □

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