Overland flow in the presence of vegetation

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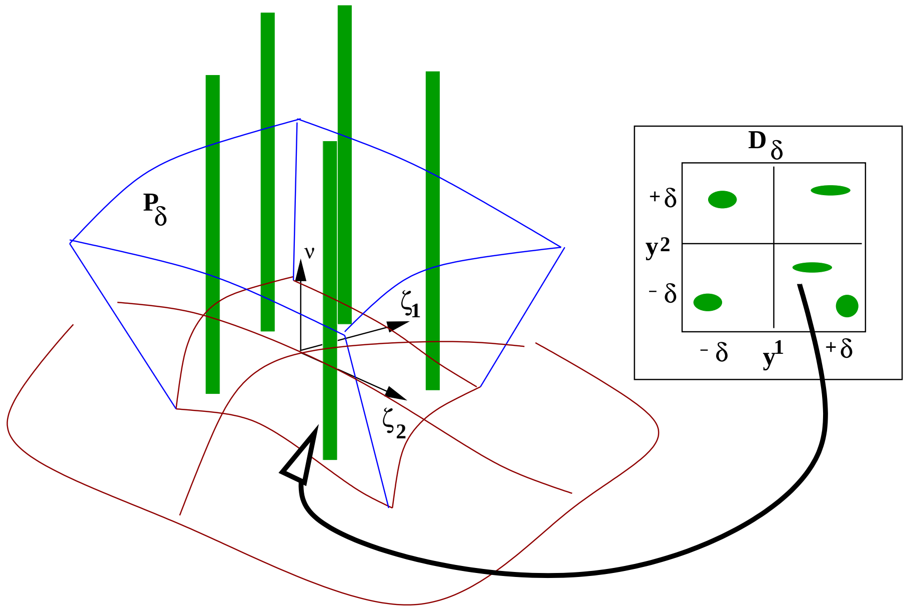

The representative element of the volume P_δ used for mediation. The bottom surface of P_δ has a representative width δ along two orthogonal directions on this surface. The water depth h associated to P_δ is the averaged value of the physical water depth $\widetilde{h}$ inside P_δ.

1 Porous Media Model

The presence of plants on the hill creates a resistance force to the water flow and influences the process of water accumulation on the soil surface. The large diversity of the growing plants on a hill makes the elaboration of an unitary model of the water flow over a soil covered by plants very difficult. Here, we present a water mass and momentum balance equations that takes into account the presence of certain type of plants.
More precisely, the plants form a dense net of rigid vertical tubes and the water fills the "voided" space up to a level not higher than these plant tubes.

1.1 Space Averaging Models

The space averaging is a method to define an unique continuous model associated to a heterogeneos fluid-solid mechanical system, the method is large used in the soil porous media models [1], [3], [9]. For the physical system fluid-plants the porous analogy was also used in [2], [4], [5], [6], [7] especially in the case of submerged vegetation.
At a hydrographic basin scale there are variations in the geometrical properties of the terrain (curvature, orientation, slope) and vegetation density or vegetation type etc. Let us assume there exists a map that models the terrain surface

$$x^i = b^i(\xi^1, \xi^2), \quad (\xi^1, \xi^2) \in D \subset \mathbb{R}^2, \text{ for i} = 1, 2, 3. \tag{1}$$

Denote the tangent vectors to the coordinate curves on this surface by

$$\varsigma_a = \partial_a \boldsymbol{b} := \frac{\partial \boldsymbol{b}}{\partial \xi^a}, \quad a = 1, 2. \tag{2}$$

Using this fixed surface, one introduces a new coordinate y^3 along the normal direction $\boldsymbol{\nu}$ to the surface. A point in the neighborhood of this surface is defined in this new system of coordinates $Y = (\xi^1, \xi^2, y^3)$ by

$$x^i = b^i(\xi^1, \xi^2) + y^3 \nu^i, \quad (\xi^1, \xi^2) \in D \subset \mathbb{R}^2,\ y^3 \in J \in \mathbb{R},\ \text{for i} = 1, 2, 3, \tag{3}$$

where $\boldsymbol{\nu} = (\nu^1, \nu^2, \nu^3)$ represents the unit normal to the surface.
We introduce the tangent vectors to the coordinate curves defined by Y,

$$\boldsymbol{\zeta}_I := \partial_I \boldsymbol{x},\ \text{for I} = 1, 2, 3. \tag{4}$$

One has

$$\boldsymbol{\zeta}_3 = \boldsymbol{\nu}, \quad \boldsymbol{\zeta}_a = (\delta_a^b - y^3 \kappa_a^b)\varsigma_b, a = 1, 2, \tag{5}$$

where $\boldsymbol{\kappa}$ is the curvature tensor of the terrain surface.
In the presence of vegetation on the hill slope the fluid occupies the free space between plant bodies and the mechanical characteristics of the fluid flow are defined only in the domain occupied by the fluid.
Notation conventions: *We adopt the following general convention: Any variable bearing a tilde over it designates micro-local physical quantity, while the absence of tilde indicates the corresponding averaged quantity. When the micro-local quantity does not differ from the corresponding averaged quantity we denote the micro-local quantity without tilde.*
Denote by Ω_f the spatial domain occupied by fluid and by Ω_p the spatial domain occupied by plants.
Consider $\widetilde{\psi}$ to be some microscopic quantity that refers to the fluid. Let $\boldsymbol{y} = (y^1, y^2)$ be a point in D, one introduces the rectangular domain

$$D_\delta = D_\delta(\boldsymbol{y}) := [y^1 - \delta, y^1 + \delta] \times [y^2 - \delta, y^2 + \delta]. \tag{6}$$

Define the spatial averaging volume

$$P = P(\boldsymbol{y}) = \{(x^1, x^2, x^3)\,\big|\, x^i = b^i(\xi^1, \xi^2) + y^3 \nu^i, \quad 0 < y^3 < \bar{h}(\xi^1, \xi^2),\ (\xi^1, \xi^2) \in D_\delta(\boldsymbol{y}),\ i = 1, 2, 3\}.$$

Here, $\bar{h}$ is some extension of $\tilde{h}$ to the domain D, where $\tilde{h}$ is the function describing the free water surface outside the domain occupied by plants.
Let us denote by P^f the fluid domain inside P,

$$P^f := P \cap \Omega^f.$$

The boundary of P^f can be partitioned as follows,

$$\partial P^f = \Sigma^{fp} \cap \Sigma^{ff} \cap \Sigma^{fa} \cap \Sigma^{fs},$$

where Σ^{fp} is the contact surface fluid-plant inside P^f, Σ^{fa} is the free surface of the fluid inside P^f, Σ^{fs} is the contact surface fluid-soil inside P^f and Σ^{ff} is the boundary surface separating the fluid inside and outside P^f.
The general form of a balance equation is, [8]

$$\partial_t \int\limits_{P^f} \widetilde{\rho}\widetilde{\psi}\mathrm{d}V + \int\limits_{\partial P^f} \widetilde{\rho}\widetilde{\psi}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n} - u_n)\mathrm{d}\sigma = \int\limits_{\partial P^f} \widetilde{\boldsymbol{\Phi}}_\psi \cdot \boldsymbol{n}\mathrm{d}\sigma + \int\limits_{P^f} \widetilde{\rho}\widetilde{\boldsymbol{\phi}}_\psi \mathrm{d}V. \tag{7}$$

Here, the significance of the above quantities are:
- $\widetilde{\rho}$ – the micro-local mass density of the fluid;
- $\widetilde{\boldsymbol{v}}$ – the micro-local velocity of the fluid;
- $\boldsymbol{n}$ – the exterior unit normal on ∂P^f;
- $\widetilde{\boldsymbol{\Phi}}_\psi$ – the micro-local flux density of $\widetilde{\psi}$;
- $\widetilde{\boldsymbol{\phi}}_\psi$ – the micro-local mass density of supply $\widetilde{\psi}$;
- u_n – the normal surface velocity;
- $\mathrm{d}V$ – the volume element;
- $\mathrm{d}\sigma$ – the surface element;

To obtain a mathematical treatable model one needs to make some assumptions concerning the complex system fluid-plant-soil. The first assumption refers to the plant cover

Assumptions 1.1 (Vegetation structure) *The plant cover satisfies:*
A1. *The plants are almost normal to the terrain surface and they behave like rigid sticks.*
A2. *The water depth is smaller than the hight of the plants.*

The assumption A1 is often used in the porous model of the vegetation and the assumption A2 is proper to the overland flow.
The soil-fluid interface $\mathcal{I}_{fs}$ and fluid-air interface $\mathcal{I}_{fa}$ can be represented as

$$\mathcal{I}_{fs} := \{\boldsymbol{x} \Big| x^i = b^i(\xi^1, \xi^2), \quad (\xi^1, \xi^2) \in D^f,\ i = 1, 2, 3\}$$

and

$$\mathcal{I}_{fa} := \{\boldsymbol{x} \Big| x^i = b^i(\xi^1, \xi^2) + \widetilde{h}(\xi^1, \xi^2)\delta_3^i, \quad (\xi^1, \xi^2) \in D^f,\ i = 1, 2, 3\},$$

respectively.
Define the averaged water depth by

$$h(y^1, y^2, t) := \frac{1}{\omega_f} \int\limits_{D_\delta^f} \widetilde{h}(\xi^1, \xi^2, t)\beta(\xi^1, \xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2, \tag{8}$$

where ω_f measures the area of Σ^{fs},

$$\omega_f := \int\limits_{D_\delta^f} \beta(\xi^1, \xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2. \tag{9}$$

The volume of the fluid inside the elementary domain P is given by

$$\mathrm{vol}(P^f) = \omega_f h. \tag{10}$$

A pure geometrical result which refers to the flux of $\widetilde{\psi}$ through the boundary Σ^{ff} is formulated as:

Lemma 1.1

$$\int_{\Sigma^{ff}} \widetilde{\rho}\widetilde{\psi}\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n} \mathrm{d}\sigma = \partial_a \int_{D^f} \int_0^{\widetilde{h}(\xi^1,\xi^2,t)} \widetilde{\rho}\widetilde{\psi}\widetilde{v}^a \Delta \mathrm{d}y^3 \beta(\xi^1,\xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2, \tag{11}$$

where $\Delta = 1 - y^3 K_M + (y^3)^2 K_G$, with K_M and K_G the mean and Gauss curvature respectively, $\beta \mathrm{d}\xi\mathrm{d}\eta$ is the area element of the terrain surface. The quantities $\widetilde{v}^a$, with $a = 1, 2$ stand for the contravariant components of the velocity fields in the local base $\{\boldsymbol{\zeta}_I\}_{I=1,3}$,

$$\widetilde{\boldsymbol{v}} = \widetilde{v}^a \boldsymbol{\zeta}_a + \widetilde{v}^3 \boldsymbol{\nu}.$$

In the (1.1) the partial differentiation ∂_a stands for

$$\partial_a := \frac{\partial}{y^a}.$$

1.2 Mass balance equation

Although the water density is considered to be a constant function, we keep it in the mass balance formulation for emphasizing the physical meaning of equations. Define the averaged water flux by

$$\rho v^a(\boldsymbol{x},t) := \frac{1}{\mathrm{vol}(P^f)} \int_{D^f_\delta} \int_0^{\widetilde{h}(\xi^1,\xi^2,t)} \widetilde{\rho}\widetilde{v}^a \Delta \mathrm{d}y^3 \beta \mathrm{d}\xi^1\mathrm{d}\xi^2. \tag{12}$$

The mass balance equation results from (7) by taking $\widetilde{\psi} = 1$, $\widetilde{\boldsymbol{\Phi}}_\psi = 0$ and $\widetilde{\phi}_\psi = 0$. Since the plants are treated as solid bodies and the water does not penetrate the plant bodies, the water flux through the boundary of the elementary volume P^f reduces to

$$\int_{\partial P^f} \widetilde{\rho}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n} - u_n)\mathrm{d}\sigma = \int_{\Sigma^{ff}} \widetilde{\rho}\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n}\mathrm{d}\sigma + \int_{\Sigma^{fa}} \widetilde{\rho}(\tilde{\boldsymbol{v}} \cdot \boldsymbol{n} - u_n)\mathrm{d}\sigma + \int_{\Sigma^{fs}} \widetilde{\rho}\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n}\mathrm{d}\sigma.$$

The second integral in r.h.s. of the above relation represents the water flux due to the rain which leads to the water mass gain inside P^f. The third term corresponds to the water flux due to the infiltration which contributes to the water loss inside P^f. Using Lemma 1.1 and the definition of the averaged quantities, one can write the mass balance:

$$\frac{\partial}{\partial t}(\omega_f h) + \partial_a (\omega_f h v^a) = \omega r - \omega_f i, \tag{13}$$

with

$$\int\limits_{\Sigma^{fa}} \widetilde{\rho}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n} - u_n)\mathrm{d}\sigma = -\rho\omega r \quad \text{and} \quad \int\limits_{\Sigma^{\mathrm{fs}}} \widetilde{\rho}\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n}\mathrm{d}\sigma = \rho\omega_{\mathrm{f}}\mathrm{i} \tag{14}$$

representing the rain rate and the infiltration rate, respectively. Here, similar as in (9), ω is defined as

$$\omega := \int_{D_\delta} \beta(\xi^1, \xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2.$$

1.3 Averaged Momentum Balance Equations

The momentum balance equation results from (7) with $\widetilde{\psi} = \widetilde{\boldsymbol{v}}$, $\widetilde{\boldsymbol{\Phi}}_\psi = \widetilde{\boldsymbol{T}}$, where $\widetilde{\boldsymbol{T}}$ is the stress tensor and $\widetilde{\phi}_\psi = \widetilde{\boldsymbol{f}}$, with $\widetilde{\boldsymbol{f}}$ denoting the body forces. Here, we only consider the gravitational force.
In contrast with the planar case, there exist some difficulties in writing component-wise the space averaging balance momentum equations. These difficulties appear due to the point dependence of the local basis. In the euclidean basis of X the momentum of the elementary volume P^f is given by

$$\mathcal{H}^i(P^f) = \int\limits_{P^f} \widetilde{\rho}\widetilde{v}^i\mathrm{d}V.$$

Using the components of $\widetilde{\boldsymbol{v}}$ in the basis of coordinates Y, we obtain

$$\mathcal{H}^i(P^f) = \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}\zeta_a^i\widetilde{v}^a\Delta\mathrm{d}y^3\mathrm{d}\sigma + \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}\nu^i\widetilde{v}^3\Delta\mathrm{d}y^3\mathrm{d}\sigma, \tag{15}$$

which can be rewritten as

$$\mathcal{H}^i(P^f) = \varsigma_a^i \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}\widetilde{v}^a\Delta\mathrm{d}y^3\mathrm{d}\sigma + \nu^i \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}\widetilde{v}^3\Delta\mathrm{d}y^3\mathrm{d}\sigma + \mathcal{E}_1^i(\widetilde{\boldsymbol{v}}, P^f). \tag{16}$$

Here, and in what follows we make the following convention: $\boldsymbol{\varsigma}_a = \boldsymbol{\varsigma}_a(\boldsymbol{y})$, where $\boldsymbol{y} = (y^1, y^2)$ is the point defining the domain $D_\delta(\boldsymbol{y})$ from (6). The unit normal $\boldsymbol{\nu}$ when appears inside the integral is a variable quantity depending on the current point from the domain D_δ, but when it appears outside the integral it is the unit normal defined by the same $\boldsymbol{y}$ as $\boldsymbol{\varsigma}_a$.
The term

$$\mathcal{E}_1^i(\widetilde{\boldsymbol{v}}, P^f) := \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}(\zeta_a^i - \varsigma_a^i)\widetilde{v}^a\Delta\mathrm{d}y^3\mathrm{d}\sigma$$

represents an error introduced by neglecting the variation of the base $\boldsymbol{\zeta}_I$ along the domain P^f.
By averaging, from (16) one has

$$\mathcal{H}(P^f) = \rho h\omega_f v^a\boldsymbol{\varsigma}_a + \rho h\omega_f v^3\boldsymbol{\nu} + \mathcal{E}_1(\widetilde{\boldsymbol{v}}, P^f). \tag{17}$$

If one neglects the momentum transfer on the fluid-air and fluid-soil intefaces the flux of the momentum through the boundary ∂P^f can be reduced to

$$\mathcal{F}(\widetilde{\rho}\widetilde{\boldsymbol{v}}, \partial P^f) := \int\limits_{\partial P^f} \widetilde{\rho}\widetilde{\boldsymbol{v}}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n} - u_n)\mathrm{d}\sigma = \int\limits_{\Sigma^{ff}} \widetilde{\rho}\widetilde{\boldsymbol{v}}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n})\mathrm{d}\sigma.$$

Using Lemma 1.1, one has

$$\mathcal{F}(\widetilde{\rho}\widetilde{\boldsymbol{v}}, \partial P^f) = \partial_a \int\limits_{D^f} \int\limits_0^{\widetilde{h}(\xi^1,\xi^2,t)} \widetilde{\rho}\widetilde{\boldsymbol{v}}\widetilde{v}^a \Delta \mathrm{d}y^3 \beta(\xi^1, \xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2,$$

and then,

$$\mathcal{F}(\widetilde{\rho}\widetilde{\boldsymbol{v}}, \partial P^f) = \partial_a(\rho\omega_f h v^b v^a \boldsymbol{\varsigma}_b) + \partial_a(\rho\omega_f h w^{ba}\boldsymbol{\varsigma}_b) + \partial_a(\rho\omega_f h v^3 v^a \boldsymbol{\nu}) + \mathcal{E}_2(\widetilde{v}^2, P^f), \tag{18}$$

where the fluctuation

$$\rho w^{ab} := \frac{1}{\omega_f h}\int_{\Sigma^f}\int_0^{\widetilde{\boldsymbol{h}}(\xi^1,\xi^2,t)} \widetilde{\rho}(\widetilde{v}^b - v^b)\widetilde{v}^a y^3 \beta(\xi^1, \xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2.$$

The quantity $\mathcal{E}_2(\widetilde{v}^2, P^f)$ (as $\mathcal{E}_1(\widetilde{v}, P^f)$ appearing above), represents the error introduced by approximating the variable local basis $(\boldsymbol{\zeta}_1(\xi^1, \xi^2, y^3), \boldsymbol{\zeta}_2(\xi^1, \xi^2, y^3), \boldsymbol{\nu}(\xi^1, \xi^2, 0))$ with the fixed local basis $(\boldsymbol{\varsigma}_1, \boldsymbol{\varsigma}_2, \boldsymbol{\nu})$ at $(y^1, y^2, 0)$. The quantities $\mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, \mathcal{E}_6$ and $\mathcal{E}_7$ introduced in what follows are errors of the same nature.
Rel. (18) can be rewritten as

$$\begin{aligned}
\mathcal{F}(\widetilde{\rho}\widetilde{\boldsymbol{v}}, \partial P^f) = \partial_a(\rho\omega_f h v^b v^a)\boldsymbol{\varsigma}_b + \rho\omega_f h v^b v^a \partial_a\boldsymbol{\varsigma}_b + \partial_a(\rho\omega_f h w^{ba})\boldsymbol{\varsigma}_b + \rho\omega_f h w^{ba}\partial_a\boldsymbol{\varsigma}_b + \\
\partial_a(\rho\omega_f h v^3 v^a)\boldsymbol{\nu} + \rho\omega_f h v^3 v^a \partial_a\boldsymbol{\nu} + \mathcal{E}_2(\widetilde{v}^2, P^f) = \\
\partial_a(\rho\omega_f h v^b v^a)\boldsymbol{\varsigma}_b + \rho\omega_f(h v^b v^a + w^{ba})(\gamma^c_{ab}\boldsymbol{\varsigma}_{\boldsymbol{c}} + \kappa_{ab}\boldsymbol{\nu}) + \partial_a(\rho\omega_f h w^{ba})\boldsymbol{\varsigma}_b + \\
\partial_a(\rho\omega_f h v^3 v^a)\boldsymbol{\nu} - \rho\omega_f h v^3 v^a \kappa^b_a\boldsymbol{\varsigma}_b + \mathcal{E}_2(\widetilde{v}^2, P^f) = \\
\partial_a(\rho\omega_f h(v^b v^a + w^{ba}))\boldsymbol{\varsigma}_b - \rho\omega_f h v^3 v^a \kappa^b_a \boldsymbol{\varsigma}_b + \rho\omega_f(h v^b v^a + w^{ba})\gamma^c_{ab}\boldsymbol{\varsigma}_{\boldsymbol{c}} + \\
\rho\omega_f(h v^b v^a + w^{ba})\kappa_{ab}\boldsymbol{\nu} + \partial_a(\rho\omega_f h v^3 v^a)\boldsymbol{\nu} + \mathcal{E}_2(\widetilde{v}^2, P^f),
\end{aligned} \tag{19}$$

where γ^c_{ab} are the Christoffel symbols.
To express the contribution of the stress forces to the momentum balance, we analyze the contribution of each interface

$$\int\limits_{\partial P^f} \widetilde{\boldsymbol{T}} \cdot \boldsymbol{n}\mathrm{d}\sigma = \mathcal{F}(\widetilde{\boldsymbol{T}}, \Sigma^{fs}) + \mathcal{F}(\widetilde{\boldsymbol{T}}, \Sigma^{fa}) + \mathcal{F}(\widetilde{\boldsymbol{T}}, \Sigma^{fp}) + \mathcal{F}(\widetilde{\boldsymbol{T}}, \Sigma^{ff}).$$

We decompose the stress tensor field $\widetilde{\boldsymbol{T}}$ in two components: the pressure field $\widetilde{p}$ and the viscous part of the stress tensor field $\widetilde{\boldsymbol{\tau}}$

$$\widetilde{\boldsymbol{T}} = -\widetilde{p}\boldsymbol{I} + \widetilde{\boldsymbol{\tau}}.$$

The pressure field is determined up to a constant value. If we subtract the atmospheric pressure from the water pressure, on the interface fluid-air the pressure must be zero. We assume the pressure field to be hydrostatically distributed.
Let $\boldsymbol{g} = -g\boldsymbol{i}_3$ be the gravitational force acting on the mass unit. In the local frame of coordinates related to the free surface of the fluid this force has the representation

$$\boldsymbol{g} = \tilde{f}^a \boldsymbol{\zeta}_a - \tilde{f}^3 \boldsymbol{\nu}.$$

Assumptions 1.2 (Hydrostatic approximation) *One assume that,*
A3. *The hydrostatic pressure field has the form,*

$$\widetilde{p}(\xi^1, \xi^2, y^3) = \tilde{\rho}\tilde{f}^3(\widetilde{h}(\xi^1, \xi^2) - y^3).$$

We neglect the shear forces on the fluid-air interface, i.e.

$$\mathcal{F}(\widetilde{\boldsymbol{T}}, \Sigma^{fa}) = 0.$$

On the fluid-soil interface

$$\mathcal{F}(\widetilde{\boldsymbol{T}}, \Sigma^{fs}) = -\int\limits_{\Sigma^{fs}} \widetilde{p}\boldsymbol{n} \mathrm{d}\sigma + \int\limits_{\Sigma^{fs}} \widetilde{\boldsymbol{\tau}} \cdot \boldsymbol{n} \mathrm{d}\sigma. \tag{20}$$

Consider the stress vector of the form $\tilde{\boldsymbol{t}} := \tilde{\boldsymbol{\tau}} \cdot \boldsymbol{n}$. In the basis $\{\boldsymbol{\zeta}_I\}$ defined by the system of local coordinates Y we write the stress vector as

$$\tilde{\boldsymbol{t}} = \tilde{t}^a \boldsymbol{\zeta}_a + \tilde{t}^3 \boldsymbol{\nu}.$$

Now, (20) takes the form

$$\mathcal{F}(\widetilde{\boldsymbol{T}}, \Sigma^{fs}) = -\boldsymbol{\nu} \int\limits_{\Sigma^{fs}} \widetilde{p} \mathrm{d}\sigma + \boldsymbol{\varsigma}_a \int\limits_{\Sigma^{fs}} \tilde{t}^a \mathrm{d}\sigma + \boldsymbol{\nu} \int\limits_{\Sigma^{fs}} \tilde{t}^3 \mathrm{d}\sigma + \mathcal{E}_3(\widetilde{\boldsymbol{T}}, \Sigma^{fs}). \tag{21}$$

Introducing the shear force at the fluid-soil interface

$$\sigma_s^a = \frac{1}{\rho \omega_f} \int\limits_{\Sigma_l^{fs}} \widetilde{t}^a \mathrm{d}\sigma,$$

(21) takes the form

$$\mathcal{F}(\widetilde{\boldsymbol{T}}, \Sigma^{fs}) = -\boldsymbol{\nu} \int\limits_{\Sigma^{fs}} \widetilde{p} \mathrm{d}\sigma + \boldsymbol{\varsigma}_a \rho \omega_f \sigma_s^a + \boldsymbol{\nu} \int\limits_{\Sigma^{fs}} \tilde{t}^3 \mathrm{d}\sigma + \mathcal{E}_3(\widetilde{\boldsymbol{T}}, \Sigma^{fs}). \tag{22}$$

On the fluid-plant interface

$$\mathcal{F}(\widetilde{\boldsymbol{T}}, \Sigma^{fp}) = -\int\limits_{\Sigma^{fp}} \widetilde{p}\boldsymbol{n} \mathrm{d}\sigma + \int\limits_{\Sigma^{fp}} \widetilde{\boldsymbol{\tau}} \cdot \boldsymbol{n} \mathrm{d}\sigma. \tag{23}$$

This interface is composed by the stem surfaces Σ_l^{fs} for each plant l inside P, i.e. $\Sigma^{fs} = \bigcup_l \Sigma_l^{fs}$.
In the hydrostatic approximation for the pressure the first integral in the r.h.s. of (23) is zero. Then, since the plant stems are supposed to be perpendicular to the ground surface, (23) becomes

$$\mathcal{F}(\widetilde{\boldsymbol{T}}, \Sigma^{fp}) = \varsigma_a \sum_l \int_{\Sigma_l^{fp}} \tilde{t}^a \mathrm{d}\sigma + \mathcal{E}_4(\widetilde{\boldsymbol{\tau}}, \Sigma^{fp}) \tag{24}$$

and introducing the plant resistance force

$$\sigma_p^a = \frac{1}{\rho} \sum_l \int_{\Sigma_l^{fp}} \tilde{t}^a \mathrm{d}\sigma,$$

(24) becomes

$$\mathcal{F}(\widetilde{\boldsymbol{T}}, \Sigma^{fp}) = \rho \sigma_p^a \varsigma_a + \mathcal{E}_4(\widetilde{\boldsymbol{\tau}}, \Sigma^{fp}). \tag{25}$$

On the fluid interface of P^f, by Lemma 1.1 for the pressure of the stress tensor, we have

$$\mathcal{F}(\widetilde{p}, \Sigma^{ff}) = \int_{\Sigma^{ff}} \widetilde{p} \boldsymbol{n} \mathrm{d}\sigma = \partial_a \int_{\Sigma^{fs}} \int_0^{\widetilde{h}} \widetilde{p} g^{ab} \boldsymbol{\zeta}_b \Delta \mathrm{d}y^3 \mathrm{d}\sigma,$$

where the g^{ab} are the contravariant components of the metric tensor.
Under the hydrostatic approximation for the pressure we can write

$$\mathcal{F}(\widetilde{p}, \Sigma^{ff}) = \frac{1}{2} \partial_a (\rho \omega_f h^2 \beta^{ab} f^3 \varsigma_b) + \mathcal{E}_5(\tilde{h}^2, \tilde{f}^3, P^f), \tag{26}$$

where β^{ab} are the contravariant components of the metric tensor of the ground surface.
From (26) we can write

$$\begin{aligned} \mathcal{F}(\widetilde{p}, \Sigma^{ff}) &= \frac{1}{2} \partial_a (\rho \omega_f h^2 \beta^{ab} f^3) \varsigma_b + \frac{1}{2} \rho \omega_f h^2 \beta^{ab} f^3 \partial_a \varsigma_b + \mathcal{E}_5(\tilde{h}^2, \tilde{f}^3, P^f) = \\ &\frac{1}{2} \partial_a (\rho \omega_f h^2 \beta^{ab} f^3) \varsigma_b + \frac{1}{2} \rho \omega_f h^2 \beta^{ab} f^3 \gamma_{ab}^c \varsigma_c + \frac{1}{2} \rho \omega_f h^2 \beta^{ab} f^3 \kappa_{ab} \boldsymbol{\nu} + \mathcal{E}_5(\tilde{h}^2, \tilde{f}^3, P^f). \end{aligned} \tag{27}$$

Invoking again Lemma 1.1, the contribution of the viscous part of the stress tensor on the interface fluid-fluid takes the form

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{ff}) = \partial_a \int_{\Sigma^{fs}} \int_0^{\widetilde{h}} \widetilde{\tau}^{ba} \boldsymbol{\zeta}_b \Delta \mathrm{d}y^3 \mathrm{d}\sigma + \partial_a \int_{\Sigma^{fs}} \int_0^{\widetilde{h}} \widetilde{\tau}^{3a} \boldsymbol{\nu} \Delta \mathrm{d}y^3 \mathrm{d}\sigma.$$

Then, we write the above quantity as,

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{ff}) = \partial_a (\omega_f h \tau^{ba} \varsigma_b) + \partial_a (\omega_f h \tau^{3a} \boldsymbol{\nu}) + \mathcal{E}_6(\widetilde{\boldsymbol{\tau}}_v, P^f). \tag{28}$$

Rel. (28) implies,

$$\begin{aligned}
&\mathcal{F}(\widetilde{\boldsymbol{\tau}},\Sigma^{ff}) = \partial_a(\omega_f h\tau^{ba})\boldsymbol{\varsigma}_b + \omega_f h\tau^{ba}\partial_a\boldsymbol{\varsigma}_b + \partial_a(\omega_f h\tau^{3a})\boldsymbol{\nu} + \omega_f h\tau^{3a}\partial_a\boldsymbol{\nu} + \mathcal{E}_6(\widetilde{\boldsymbol{\tau}}_v, P^f) = \\
&\partial_a(\omega_f h\tau^{ba})\boldsymbol{\varsigma}_b + \omega_f h\tau^{ba}(\gamma^c_{ab}\boldsymbol{\varsigma}_c + \kappa_{ab}\boldsymbol{\nu}) + \partial_a(\omega_f h\tau^{3a})\boldsymbol{\nu} - \omega_f h\tau^{3a}\kappa^b_a\boldsymbol{\varsigma}_b + \mathcal{E}_6(\widetilde{\boldsymbol{\tau}}_v, P^f) = \\
&\partial_a(\omega_f h\tau^{ba})\boldsymbol{\varsigma}_b - \omega_f h\tau^{3a}\kappa^b_a\boldsymbol{\varsigma}_b + \omega_f h\tau^{ba}\gamma^c_{ab}\boldsymbol{\varsigma}_c + \omega_f h\tau^{ba}\kappa_{ab}\boldsymbol{\nu} + \partial_a(\omega_f h\tau^{3a})\boldsymbol{\nu} + \mathcal{E}_6(\widetilde{\boldsymbol{\tau}}_v, P^f).
\end{aligned} \tag{29}$$

For the supply $\tilde{\psi}$ we only consider the contribution of the gravitational force. Proceeding by components as in (16), the second term in the r.h.s. of (7) is finally expressed as

$$\int_{P^f} \widetilde{\rho}\widetilde{\boldsymbol{\phi}}_\psi \mathrm{d}V = \int_{P^f} \widetilde{\rho}\tilde{f}^a\boldsymbol{\zeta}_a \mathrm{d}V - \int_{P^f} \widetilde{\rho}\tilde{f}^3\boldsymbol{\nu}\mathrm{d}V = \rho h\omega_f f^a\boldsymbol{\varsigma}_a - \rho h\omega_f f^3\boldsymbol{\nu} + \mathcal{E}_7(\tilde{f}^1, \tilde{f}^2, \tilde{f}^3, P^f). \tag{30}$$

The relations (17,19,22,25,27,29) and some order assumptions are the base for averaged momentum equations. Let ϵ be a small parameter.

Assumptions 1.3 (Kinematical and topographical assumptions) *Suppose that the physical processes satisfy the following properties:*
A4. The water depth. $\widetilde{h} = O(\epsilon)$.
A5. The velocity. $v^3 = O(\epsilon)$.
A6. Curvature. *The terrain surface curvatures and the curvature of the coordinate curves are of order of ϵ. This means that locally the surface is almost planar.*
A7. The averaged dimension δ. $d_p \ll \delta \ll L$ *and* $\delta K_M = O(\epsilon)$.

The shalow water type approximation of the averaged momentum balance for an incompressible fluid results by performing an asymptotic analysis.

Theorem 1.1 (Averaged momentum equations) *If the assumptions* A1–A7 *are in force then the first order approximation of the momentum equations are given by*

$$\partial_t(h\beta\theta v^a) + \partial_b\mathfrak{F}^{ab}(h,v) = \mathfrak{G}^a(h,v), \quad a = 1,2, \tag{31}$$

where

$$\mathfrak{F}^{ab}(h,v) = h\beta\theta\left(v^a v^b + \frac{h}{2}\beta^{ab}f^3 + w^{ab} - \frac{1}{\rho}\tau_v^{ab}\right),$$

$$\mathfrak{G}^a(h,v) = h\beta\theta f^a + \sigma^a_p + \beta\theta\sigma^a_s - \gamma^a_{bc}\eta^{bc}$$

and

$$\eta^{ac} = h\beta\theta\left(v^a v^b + \frac{h}{2}\beta^{ab}f^3 + w^{ab} - \frac{1}{\rho}\tau_v^{ab}\right).$$

The θ function denotes the porosity of the plant cover and is defined by

$$\theta = \frac{\omega_f}{\omega}$$

Sketch of proof. Using Assumptions 1.3 and relations (17,19,22,25,27,29) one can prove that the terms $\mathcal{E}_1, \ldots, \mathcal{E}_7$ are of order ϵ^2. For $\epsilon \ll 1$ these terms as well as the terms containing the factors $v^3 h$, $h\kappa$ or h^2 (which are of same order ϵ^2) can be neglected.

The equations (31) must be supplemented by empirical laws concerning the *averaged stress tensor* $\boldsymbol{\tau}$, the *averaged vegetation force resistance* $\boldsymbol{\sigma}_p$, the *averaged shear fluid-soil force* $\boldsymbol{\sigma}_s$ and the *averaged fluctuation* w^{ab}. These empirical laws are expressed by functions depending on the averaged velocity $\boldsymbol{v}$, the averaged water depth h and a set of parameters $\boldsymbol{\lambda}$ defined by the characteristics of the plant cover.

$$\begin{cases} \tau_v^{ab} = \mathfrak{T}^{ab}(\nabla \boldsymbol{v}, h, \boldsymbol{\lambda}), \\ \sigma_p^b = \mathfrak{S}_p^b(\boldsymbol{v}, h, \boldsymbol{\lambda}), \\ \sigma_s^b = \mathfrak{S}_s^b(\boldsymbol{v}, h, \boldsymbol{\lambda}), \\ w^{ab} = \mathfrak{W}^{ab}(\boldsymbol{v}, h, \boldsymbol{\lambda}). \end{cases} \tag{32}$$

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