Solving Generalized 2D Navier-Stokes Equations

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Contents

Abstract

In this paper we set up a numerical algorithm for computing the flow of a class of pseudo-plastic fluids. Such a model, with the viscosity depending on the strain rate, is frequently used as a mechanical model of the blood flow. The method uses the finite volume technique for space discretization and a semi-implicit two steps backward differentiation formula for time integration. As primitive variables the algorithm

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uses the velocity field and pressure field. In this scheme quadrilateral structured primal-dual meshes are used. The velocity and the pressure fields are discretized on the primal mesh and the dual mesh respectively. By a proper definition of the discrete derivative operators we were able to prove a Hodge decomposition formula. Based on it we can calculate independently the velocity and pressure. A certain advantage of the method is that the velocity and pressure can be computed without any artificial boundary conditions and initial data for the pressure. Based on the numerical algorithm we have written a numerical code. We have also performed a series of numerical simulations.

1 Introduction

In this paper we are interested in the numerical approximation of a class of pseudo-plastic fluid flow. The motion of the fluid is described by the generalized incompressible Navier-Stokes equations

$$\left\{ \begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= -\nabla p + \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} &= 0, \end{aligned} \right. \tag{1}$$

where $\boldsymbol{u}$ is the velocity vector field, p is the hydrodynamic pressure field, $\boldsymbol{\sigma}$ the extra stress tensor field and $\boldsymbol{f}$ is the body force. The extra stress tensor $\boldsymbol{\sigma}(\boldsymbol{u})$ obey a constitutive equation of the type

$$\sigma_{ab}(\widetilde{\partial u}) = 2\nu(|\widetilde{\partial u}|)\widetilde{\partial u}_{ab} \tag{2}$$

where $\widetilde{\partial u}$ is the strain rate tensor given by

$$\widetilde{\partial u}_{ab} = \frac{1}{2}\left(\partial_a u_b + \partial_b u_a\right),$$

∂_a stands for the partial derivative with respect with space coordinate x_a, and for any square matrix $\boldsymbol{e}$, $|\boldsymbol{e}|$ is definedas

$$|\boldsymbol{e}| = \left(\sum_{i,j} e_{ij}^2 \right)^{1/2}.$$

Concerning the viscosity function $\nu(s)$, we assume that it is a continuous differentiable, decreasing function, with bounded range

$$\left\{ \begin{array}{l} 0 < \nu_\infty \le \nu(s) \le \nu_0 < \infty, \forall s > 0, \\ \left(\nu(s_1) - \nu(s_2)\right)(s_1 - s_2) < 0, \forall s_1, s_2 > 0, \end{array} \right. \tag{3}$$

and it satisfies the constraint

$$\nu(s) + s\dot{\nu}(s) > c > 0. \tag{4}$$

The model of the Newtonian fluid corresponds to ν constant.

We consider the case when the flow take place inside a fixed and bounded domain $\Omega \in \mathbb{R}^2$ and we assume that the fluid adheres to its boundary $\partial\Omega$, hance we impose a Dirichlet type boundary conditions for the velocity field

$$\boldsymbol{u} = \boldsymbol{u}_D(x),\ x \in \partial\Omega, t > 0. \tag{5}$$

To the equations (1) we append the initial condition for the velocity

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}_0(x),\ x \in \Omega. \tag{6}$$

The initial boundary value problem (IBV), which we intend to solve numerically, consists in finding the velocity field $\boldsymbol{u}(x,t)$ and the pressure field $p(x,t)$ that satisfy the partial differential equations (1), boundary condition (5) and the initial condition (6).

In writing down a numerical algorithm for the non-stationary incompressible generalized Navier-Stokes equations there exists three main difficulties, namely: (i) the velocity field and the pressure field are coupled by the incompressibility constraint [17], (ii) the presences of the nonlinear convection term and (iii) the nonlinear dependences of the viscosity on the share rate.

The first two problems are common to the Navier-Stokes equations and in the last fifty years several methods was developed to overcome them, the projection method, the artificial compressibility method and gauge method to mention the most significants for our case.

In the projection-type algorithms the computations of the velocity and the pressure are decoupled, see [17] for an overview of the projection methods. The numerical method developed by Harlow and Welch [18], attempts to enforce the incompressibility constraint by deriving a Poisson equation for the pressure, taking the divergence of the momentum balance equation. Such kind of the methods needs an artificial boundary conditions for the pressure.

Chorin [11] developed a practical numerical method based on a discrete form of the Hodge decomposition. This method, known as the projection method, computes an intermediate vector field that is then projected onto divergence free fields to recover the velocity.

Kim and Moin [19] proposed a method for advancing velocity field in two fractional steps. They use a version of Chorin's algorithm replacing the treatment of nonlinear convective term with second order explicit Adams-Bashford scheme and the implicit second-order Crank-Nicholson method for viscous term. The divergent-free velocity is updated by using a projection function that solve a Poisson equation.

Bell, Colella and Glaz, [3] developed a second order accurate method in time and space. The method is a kind of Crank-Nicholson method for time stepping and uses an intermediate values for velocity field and pressure field.

The gauge method, introduced by Weinan E and Jianguo Liu [27], works with a gauge variable ϕ and a vectorial field $\boldsymbol{a}$, the velocity $\boldsymbol{u}$ is related to the new variables by $\boldsymbol{a} = \boldsymbol{u} + \nabla\phi$. Knowing the gauge field ϕ one can computes the pressure. As authors said the main advantage of the method is that one can use "the gauge freedom to assign an unambiguous boundary condition for $\boldsymbol{a}$ and ϕ".

When one deal with non-Newtonian fluid the nonlinearity of the viscosity rise a new problem in obtaining a discrete form a Navier-Stokes equations. The new issue is the development of an appropriate discrete form of the action of the stress tensor on the boundary of the volume-control. In fact the problem is how one define the gradient of the discrete velocity field on the boundary of the finite volume. Andreianov, Boyer and Hubert [2] develop a method for solving p-laplacian problem for rectangular grid. They use primal and dual mesh and define the flux on the boundary of the control volume. The discrete gradient of the unknown is defined on the dual mesh and the norm of the gradient on the boundary of the control volume is evaluated as quadratic bilinear function on the gradient defined on dual mesh.

The outline of the paper is as follows. In the section 2 we present the functional analysis frame concerning the weak solution of IBV (1), (5 and (6) and we prove the existence of it for a class of pseudo-plastic fluids which satisfy (4). In the section 3 we establish the semi-discrete, space discrete coordinates and continuum time variable, form of the equation (1) and we present some general concepts concerning the space discretisation and related notions like admissible mesh, primal and dual mesh, the discretisation of the derivative operators etc. In the section 4 we present an algorithm to solve a 2D model. Basically, the method computes the co-ordinates of the velocity field in a base of discrete solenoidal vector fields. In the last section we perform some numerical simulation of the lid driven cavity flow.

2 The Existence of the Weak Solution

We introduce some notations. By $L^p(\Omega)$ and $W^{m,p}(\Omega)$, $m = 0, 1, \cdots$, we denote the usual Lebesgue and Sobolev spaces, respectively. The scalar product in L^2 is indicated by $(\cdot, \cdot)$. For $\boldsymbol{u}$, $\boldsymbol{v}$ vector functions defined on Ω we put

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} u^a v_a \mathrm{d}x$$

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = \sum_{a,b=1} \int_{\Omega} \partial_a u^b \partial_a v^b \mathrm{d}x$$

We denote $||\cdot||$ the norm in L^2 associate to $(\cdot, \cdot)$. The norm in $W^{m,p}$ is indicated by $||\cdot||_{m,p}$. Consider the space.

$$\boldsymbol{\mathcal{V}} = \{\boldsymbol{\psi} \in C_0^\infty(\Omega), \mathrm{div}\boldsymbol{\psi} = 0\} \tag{7}$$

We define $\boldsymbol{H}(\Omega)$ as the completion of $\boldsymbol{\mathcal{V}}$ in the space $\boldsymbol{L}^2(\Omega)$. We denote by $\boldsymbol{H}^1(\Omega)$ the completion of $\boldsymbol{\mathcal{V}}$ in the space $\boldsymbol{W}^{1,2}$.

For $T \in (0, \infty]$ we set $Q_T = \Omega \times [0, T)$ and define

$$\boldsymbol{\mathcal{V}}_T = \{\phi \in \boldsymbol{C}_0^\infty(Q_T);\ \mathrm{div}\boldsymbol{\phi}(x,t) = 0 \text{ in } Q_T\}$$

Any function $\phi \in \boldsymbol{\mathcal{V}}_T$ can be approximate by some finite linear combination of the functions from $\boldsymbol{\mathcal{V}}$ of the type

$$\phi_N(x,t) = \sum_{l=1}^{N} \gamma_l(t)\psi_l(x) \tag{8}$$

where $\gamma_l \in C^1([0,T))$ and $\psi_l \in \boldsymbol{\mathcal{V}}$ More precise we have [**?**]

Lemma 2.1 *There exists a sequences of functions* $\{\boldsymbol{\psi}_\alpha\}_{\alpha=1}^\infty \in \boldsymbol{\mathcal{V}}$ *which constitutes an orthonormal bassi of* $\boldsymbol{H}(\Omega)$ *with the following properties. Given* $\phi \in \boldsymbol{\mathcal{V}}_T$ *and* $\epsilon > 0$ *there are* $N = N(\phi, \epsilon)$ *functions* $\gamma_\alpha(t) \in C^1([0,T))$ *such that*

$$\max_{t\in[0,T)} ||\phi_N(t) - \phi(t)||_{C^2(\Omega)} + \max_{t\in[0,T)} ||\partial_t\phi_N(t) - \partial_t\phi(t)||_{C^0(\Omega)} < \epsilon$$

where ϕ_N *is given by (8).*

The weak solution of IBV is defined as follows.

Definition 2.1 *Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$. Let $\boldsymbol{u}_0(x) \in \boldsymbol{L}^2(\Omega)$ and u_D be such that*

$$\begin{cases} \operatorname{div} \boldsymbol{u}_0 = 0, \\ \boldsymbol{u}_D \cdot \boldsymbol{n} = 0, x \in \partial\Omega, \\ \boldsymbol{u}_0 = \boldsymbol{u}_D,\ x \in \partial\Omega. \end{cases} \tag{9}$$

and there exists $\boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega) \cap \boldsymbol{L}^4(\Omega)$ a vector function which satisfies

$$\begin{cases} \operatorname{div} \boldsymbol{v} = 0, \\ \boldsymbol{v} = \boldsymbol{u}_D,\ \ x \in \partial\Omega \end{cases} \tag{10}$$

Then $\boldsymbol{u}$ is weak solution of IBV (1,5,6) *if*

$$\boldsymbol{u} - \boldsymbol{v} \in L^2\left((0,T); \boldsymbol{H}^1(\Omega)\right) \cap L^\infty\left((0,T); \boldsymbol{H}(\Omega)\right) \tag{11}$$

and $\boldsymbol{u}$ verifies

$$\begin{aligned} -\int_0^\infty \left(\boldsymbol{u}, \frac{\partial \boldsymbol{\phi}}{\partial t}\right) \mathrm{d}t - \int_0^\infty (\boldsymbol{u} \otimes \boldsymbol{u}, \nabla \boldsymbol{\phi}) \, \mathrm{d}t + \int_0^\infty \left(\sigma(\boldsymbol{u}), \widetilde{\partial \boldsymbol{\phi}}\right) \mathrm{d}t = \\ \int_0^\infty (\boldsymbol{f}, \boldsymbol{\phi}) \, \mathrm{d}t + (u_0, \phi) \end{aligned} \tag{12}$$

for any test function $\boldsymbol{\phi} \in \boldsymbol{\mathcal{V}}_T$.

It is easy to show that if u is a weak solution in Q_T then we have

Lemma 2.2 *Let u be a weak solution in Q_T. Then u satisfies:*

(a)

$$\begin{aligned} -\int_s^t \left(\boldsymbol{u}, \frac{\partial \boldsymbol{\phi}}{\partial t}\right) \mathrm{d}t - \int_s^t (\boldsymbol{u} \otimes \boldsymbol{u}, \nabla \boldsymbol{\phi}) \, \mathrm{d}t + \int_s^t \left(\sigma(\boldsymbol{u}), \widetilde{\partial \boldsymbol{\phi}}\right) \mathrm{d}t = \\ \int_s^t (\boldsymbol{f}, \boldsymbol{\phi}) \, \mathrm{d}t - (u(t), \phi(t)) + (u(s), \phi(s)) \end{aligned} \tag{13}$$

for all $\phi \in \boldsymbol{\mathcal{V}}_t$ and almost all $t, s \in [0, T)$.

(b)

$$\begin{aligned} -\int_0^t (\boldsymbol{u} \otimes \boldsymbol{u}, \nabla \boldsymbol{\psi}) \, \mathrm{d}t + \int_0^t \left(\sigma(\boldsymbol{u}), \widetilde{\partial \boldsymbol{\phi}}\right) \mathrm{d}t = \\ \int_0^t (\boldsymbol{f}, \boldsymbol{\psi}) \, \mathrm{d}t - (u(t), \psi) + (u(0), \psi) \end{aligned} \tag{14}$$

for all $\psi \in \boldsymbol{\mathcal{V}}$ and almost all $t \in [0, T)$.

In the next lemma we remaind some well known inequalities that will be used later in the paper.

Lemma 2.3 *Let* $\boldsymbol{u} \in \boldsymbol{L}^{p_1}$, $\boldsymbol{w} \in \boldsymbol{L}^{p_3}$, $\boldsymbol{v} \in \boldsymbol{W}^{1,p_2}$ *and*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$$

then

$$|(\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{w})| \leq ||\boldsymbol{u}||_{p_1} ||\boldsymbol{w}||_{p_3} ||\nabla \boldsymbol{v}||_{p_2} \tag{15}$$

For any function u *in* $\boldsymbol{u} \in \boldsymbol{W}_0^{1,2}(\Omega)$ *the inequality*

$$\int_\Omega |u|^4 \mathrm{d}x \leq 2 ||\boldsymbol{u}||^2 ||\nabla \boldsymbol{u}||^2 \tag{16}$$

holds. This inequality implies the inequality

$$||\boldsymbol{u}||_4 \leq \epsilon ||\boldsymbol{u}|| + \frac{1}{\epsilon} ||\nabla \boldsymbol{u}|| \tag{17}$$

which is true for any $\epsilon > 0$.

The Korn inequality

$$||\nabla \boldsymbol{u}|| \leq K_\Omega ||\widetilde{\partial u}|| \tag{18}$$

holds for any $u \in \boldsymbol{W}_0^{1,2}(\Omega)$.

Based on the relations (2), (3) and (4) one can prove the following lemma.

Lemma 2.4 *The extra stress tensor satisfies the following relations*

$$\begin{cases} \boldsymbol{\sigma}(\boldsymbol{d}) \cdot \boldsymbol{d} \geq c_1 |\boldsymbol{d}|^2, c_1 > 0, \\ |\boldsymbol{\sigma}(\boldsymbol{d})| \leq c_2 |\boldsymbol{d}|, c_2 > 0, \\ (\boldsymbol{\sigma}(\boldsymbol{d}) - \boldsymbol{\sigma}(\boldsymbol{e}), \boldsymbol{d} - \boldsymbol{e}) \geq c |\boldsymbol{d} - \boldsymbol{e}|^2, \end{cases} \tag{19}$$

for all symmetric tensors $\boldsymbol{d}, \boldsymbol{e} \in \mathbb{R}^{2\times 2}$

Proof. The first two relations are simple consequences of the boundedness of the viscosity function. To prove the third relation consider the real functions

$$f_{ij}(s) = \nu(|(1-s)\boldsymbol{e} + s\boldsymbol{d}|)((1-s)e_{ij} + sd_{ij})$$

for any i, j. It follows that

$$f_{ij}(1) = \sigma_{ij}(\boldsymbol{d}),\ f_{ij}(0) = \sigma_{ij}(\boldsymbol{e})$$

and

$$(\boldsymbol{\sigma}(\boldsymbol{d}) - \boldsymbol{\sigma}(\boldsymbol{e}), \boldsymbol{d} - \boldsymbol{e}) = \sum_{i,j} (f_{ij}(1) - f_{ij}(0))(d^{ij} - e^{ij})$$

On the other hand we have

$$\begin{aligned} f_{ij}(1) - f_{ij}(0) &= \int_0^1 \frac{\mathrm{d}f_{ij}(s)}{\mathrm{d}s} \mathrm{d}s \\ &= \int_0^1 \frac{\dot{\nu}(|\psi|)}{|\psi|} (\psi, \boldsymbol{d} - \boldsymbol{e})\, \psi_{ij} + \nu(|\psi|)(d_{ij} - e_{ij}) \mathrm{d}s \end{aligned}$$

where $\psi = (1 - s)\boldsymbol{e} + s\boldsymbol{d}$. Then

$$\sum_{i,j} (f_{ij}(1) - f_{ij}(0))(d^{ij} - e^{ij}) = \int_0^1 \frac{\dot{\nu}(|\psi|)}{|\psi|} (\psi, \boldsymbol{d} - \boldsymbol{e})^2 + \nu(|\psi|)|\boldsymbol{d} - \boldsymbol{e}|^2 \mathrm{d}s. \quad (20)$$

Taking into account that

$$(\psi, \boldsymbol{d} - \boldsymbol{e})^2 \leq |\psi|^2 |\boldsymbol{d} - \boldsymbol{e}|^2,$$

$\dot{\nu}(s) \leq 0$, and (4) one obtains

$$\sum_{i,j} (f_{ij}(1) - f_{ij}(0))(d^{ij} - e^{ij}) \geq \int_0^1 ((\dot{\nu}(|\psi|)|\psi| + \nu(|\psi|))|\boldsymbol{d} - \boldsymbol{e}|^2 \mathrm{d}s > c|\boldsymbol{d} - \boldsymbol{e}|^2$$

Remark. From relation (20) results that for the monotone increasing viscosity function and bounded below the third relation in (19) it is true for any supplementary conditions.

Theorem 2.1 *If the constitutive function* νs *satisfies the relations* (3) *and* (4) *then there exists a weak solution of the* IBV (1), (5 and (6)*.*

Proof. We follows the main ideas of [20].

Let $\{\boldsymbol{\psi}_\alpha\}_{\alpha=1}^{\infty}$ be the basis of $\boldsymbol{H}(\Omega)$ given in the Lemma 2.1. We search a solution of the form

$$\boldsymbol{u}(x, t) = \boldsymbol{w}(x, t) + \boldsymbol{v}(x)$$

with $\boldsymbol{w}(x, t) \in \boldsymbol{H}^1(\Omega)$ for any $t \in (0, T)$. The new unknown function will be determined as weak solution of the equation

$$\frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{w} + \boldsymbol{v}) \cdot \nabla(\boldsymbol{w} + \boldsymbol{v}) = -\nabla p + \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{w} + \boldsymbol{v}) + \boldsymbol{f}, \quad (21)$$

with the initial condition

$$\boldsymbol{w}(x,0)=\boldsymbol{w}_0(x):=\boldsymbol{u}_0(x)-\boldsymbol{v}(x),\ x\in\Omega. \tag{22}$$

We develop w in the base $\{\boldsymbol{\psi}_\alpha\}_{\alpha=1}^{\infty}$

$$\boldsymbol{w}(x,t)=\sum_{\alpha=1}^{\infty}c_\alpha(t)\boldsymbol{\psi}_\alpha$$

and let

$$\overset{n}{\boldsymbol{w}}=\sum_{\alpha=1}^{n}c_\alpha(t)\boldsymbol{\psi}_\alpha.$$

The approximation $\overset{n}{\boldsymbol{w}}$ will be calculated by solving the ordinary differential equations

$$\left(\partial_t\overset{n}{\boldsymbol{w}},\boldsymbol{\psi}_\alpha\right)-\left((\overset{n}{\boldsymbol{w}}+\boldsymbol{v})\otimes(\overset{n}{\boldsymbol{w}}+\boldsymbol{v}),\nabla\psi_\alpha\right)+\left(\boldsymbol{\sigma}(\overset{n}{\boldsymbol{w}}+\boldsymbol{v}),\widetilde{\partial\psi}_\alpha\right)=(\boldsymbol{f},\boldsymbol{\psi}_\alpha) \tag{23}$$

with initial conditions

$$\left(\overset{n}{\boldsymbol{w}},\psi_\alpha\right)=(\boldsymbol{w}_0,\psi_\alpha)\,,\alpha=\overline{1,n} \tag{24}$$

A priori estimations. By multiplication of each α-equation in (23) with c_α and summing up one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|\left|\overset{n}{\boldsymbol{w}}\right|\right|^2+\left(\left(\overset{n}{\boldsymbol{w}}+\boldsymbol{v}\right)\cdot\nabla\left(\overset{n}{\boldsymbol{w}}+\boldsymbol{v}\right),\overset{n}{\boldsymbol{w}}\right)+\left(\boldsymbol{\sigma}(\overset{n}{\boldsymbol{w}}+\boldsymbol{v}),\widetilde{\partial\overset{n}{\boldsymbol{w}}}\right)=\left(\boldsymbol{f},\overset{n}{\boldsymbol{w}}\right). \tag{25}$$

We estimate the terms in (25) separately. One obtains:

$$\begin{aligned}\left|\left(\left(\overset{n}{\boldsymbol{w}}+\boldsymbol{v}\right)\cdot\nabla\left(\overset{n}{\boldsymbol{w}}+\boldsymbol{v}\right),\overset{n}{\boldsymbol{w}}\right)\right| &=\left|\left(\left(\overset{n}{\boldsymbol{w}}+\boldsymbol{v}\right)\cdot\nabla\boldsymbol{v},\overset{n}{\boldsymbol{w}}\right)\right|=\\ &\left|-\left(\overset{n}{\boldsymbol{w}}\cdot\nabla\overset{n}{\boldsymbol{w}},\boldsymbol{v}\right)+\left(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\overset{n}{\boldsymbol{w}}\right)\right|\le\\ &\le\left|\left(\overset{n}{\boldsymbol{w}}\cdot\nabla\overset{n}{\boldsymbol{w}},\boldsymbol{v}\right)\right|+\left|\left(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\overset{n}{\boldsymbol{w}}\right)\right|\le\\ &\le||\boldsymbol{v}||_4\left(||\nabla\overset{n}{\boldsymbol{w}}||\,||\overset{n}{\boldsymbol{w}}||_4+||\nabla\boldsymbol{v}||\,||\overset{n}{\boldsymbol{w}}||_4\right)\le\\ &\le||\boldsymbol{v}||_4\left(\epsilon||\nabla\overset{n}{\boldsymbol{w}}||^2+C_1(\epsilon)||\overset{n}{\boldsymbol{w}}||^2+C_2(\epsilon)\,||\nabla\boldsymbol{v}||^2\right),\end{aligned} \tag{26}$$

$$\begin{aligned}\left(\boldsymbol{\sigma}(\overset{n}{\boldsymbol{w}}+\boldsymbol{v}),\widetilde{\partial\overset{n}{\boldsymbol{w}}}\right) &=\left(\nu(|\overset{n}{\boldsymbol{w}}+\boldsymbol{v}|)\widetilde{\partial\overset{n}{\boldsymbol{w}}},\widetilde{\partial\overset{n}{\boldsymbol{w}}}\right)+\left(\nu(|\overset{n}{\boldsymbol{w}}+\boldsymbol{v}|)\widetilde{\partial\boldsymbol{v}},\widetilde{\partial\overset{n}{\boldsymbol{w}}}\right)\ge\\ &\ge\nu_\infty||\widetilde{\partial\overset{n}{\boldsymbol{w}}}||^2-\nu_0||\widetilde{\partial\overset{n}{\boldsymbol{w}}}||\,||\widetilde{\partial\boldsymbol{v}}||\ge\\ &\ge\frac{\nu_\infty}{2}||\widetilde{\partial\overset{n}{\boldsymbol{w}}}||^2-\frac{\nu_0^2}{2\nu_\infty}||\widetilde{\partial\boldsymbol{v}}||^2,\end{aligned} \tag{27}$$

$$\left(\boldsymbol{f}, \overset{n}{\boldsymbol{w}}\right) \leq ||\boldsymbol{f}|| \, || \overset{n}{\boldsymbol{w}} || \leq \frac{1}{2} || \overset{n}{\boldsymbol{w}} ||^2 + \frac{1}{2} \, ||\boldsymbol{f}||^2 \tag{28}$$

By using (26), (27) and (28) the equality (25) becomes

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left|\left|\overset{n}{\boldsymbol{w}}\right|\right|^2 + \frac{\nu_\infty}{2} ||\widetilde{\partial \overset{n}{\boldsymbol{w}}}||^2 \quad &\leq ||\boldsymbol{v}||_4 \left(\epsilon ||\nabla \overset{n}{\boldsymbol{w}} ||^2 + C_1(\epsilon) || \overset{n}{\boldsymbol{w}} ||^2 + C_2(\epsilon) \, ||\nabla \boldsymbol{v}||^2\right) + \\ &+ \frac{1}{2} || \overset{n}{\boldsymbol{w}} ||^2 + \frac{1}{2} \, ||\boldsymbol{f}||^2 + \frac{\nu_0^2}{2\nu_\infty} ||\widetilde{\partial \boldsymbol{v}}||^2 \end{aligned}$$

where $C_1(\epsilon) = 7/\epsilon^3$, $C_2(\epsilon) = 13\epsilon/4$.

Then by using Korn inequality and taking $\epsilon = \dfrac{\nu_\infty K_\Omega^2}{4\,||\boldsymbol{v}||_4}$ one obtains

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left|\left|\overset{n}{\boldsymbol{w}}\right|\right|^2 + \frac{\nu_\infty K_\Omega^2}{4} ||\nabla \overset{n}{\boldsymbol{w}}||^2 \leq k_1 || \overset{n}{\boldsymbol{w}} ||^2 + k_2. \tag{29}$$

where the constants k_1 and k_2 depend on Ω, $\boldsymbol{f}$ and $\boldsymbol{v}$ and do not depend on n. The inequality (29) implies the inequalities:

$$|| \overset{n}{\boldsymbol{w}} (t)||^2 \leq -\frac{k_2}{k1} + \left(||\boldsymbol{w}_0||^2 + \frac{k_2}{k_1}\right) \mathrm{e}^{k_1 t} \tag{30}$$

and

$$|| \overset{n}{\boldsymbol{w}} (t)||^2 + \frac{\nu_\infty K_\Omega^2}{2} \int\limits_0^t ||\nabla \overset{n}{\boldsymbol{w}} (s)||^2 \mathrm{d}s \leq ||\boldsymbol{w}_0||^2 + \left(k_1 ||\boldsymbol{w}_0||^2 + k_2\right) \frac{\mathrm{e}^{2k_1 t} - 1}{2k_1}. \tag{31}$$

By using (19-2) one can also prove

$$|| \overset{n}{\boldsymbol{w}} (t)||^2 + c_1 \int\limits_0^t ||\sigma(\overset{n}{\boldsymbol{w}} (s))||^2 \mathrm{d}s \leq ||\boldsymbol{w}_0||^2 + \left(k_1 ||\boldsymbol{w}_0||^2 + k_2\right) \frac{\mathrm{e}^{2k_1 t} - 1}{2k_1}. \tag{32}$$

Next we show that the $g_\alpha^n(t) = \left(\overset{n}{\boldsymbol{w}}, \boldsymbol{\psi}_\alpha\right)$ form a uniform bounded and echicontinous family of real functions on $[0, T]$:

$$|g_\alpha^n(t) - g_\alpha^n(s)| = \left| \int\limits_s^t \left(\partial_t \overset{n}{\boldsymbol{w}}, \boldsymbol{\psi}_\alpha\right) \mathrm{d}t \right| \leq$$

$$\leq \int\limits_s^t \left| \left((\overset{n}{\boldsymbol{w}} + \boldsymbol{v}) \otimes (\overset{n}{\boldsymbol{w}} + \boldsymbol{v}), \nabla \boldsymbol{\psi}_\alpha \right) \right| + \left| \left(\boldsymbol{\sigma}(\overset{n}{\boldsymbol{w}} + \boldsymbol{v}), \widetilde{\partial \boldsymbol{\psi}_\alpha} \right) \right| + |(\boldsymbol{f}, \boldsymbol{\psi}_\alpha)| \, \mathrm{d}t,$$

and

$$\begin{aligned}\left|\left((\overset{n}{\boldsymbol{w}}+\boldsymbol{v})\otimes(\overset{n}{\boldsymbol{w}}+\boldsymbol{v}),\nabla\boldsymbol{\psi}_\alpha\right)\right| &\le ||\overset{n}{\boldsymbol{w}}+\boldsymbol{v}||_4^2\,||\nabla\boldsymbol{\psi}_\alpha||\le\\ &\le ||\left(\overset{n}{\boldsymbol{w}}||_4^2+||\boldsymbol{v}||_4^2\right)||\nabla\boldsymbol{\psi}_\alpha||\le\\ &\le ||\overset{n}{\boldsymbol{w}}||\,||\nabla\overset{n}{\boldsymbol{w}}||\,||\nabla\boldsymbol{\psi}_\alpha||+||\boldsymbol{v}||_4^2||\nabla\boldsymbol{\psi}_\alpha||\le\\ &\le k_1(\alpha,T)\,||\nabla\overset{n}{\boldsymbol{w}}||+k_2(\alpha),\end{aligned}$$

$$\begin{aligned}\left|\left(\boldsymbol{\sigma}(\overset{n}{\boldsymbol{w}}+\boldsymbol{v}),\widetilde{\partial\boldsymbol{\psi}_\alpha}\right)\right| &\le \nu_0||\widetilde{\partial\overset{n}{\boldsymbol{w}}}+\widetilde{\partial\boldsymbol{v}}||\,||\widetilde{\partial\boldsymbol{\psi}_\alpha}||\le\\ &\le k_3(\alpha)||\nabla\overset{n}{\boldsymbol{w}}||+k_4(\alpha),\end{aligned}$$

$$|(\boldsymbol{f},\boldsymbol{\psi}_\alpha)|\le k_5(\alpha)||f||.$$

Then we obtain

$$\begin{aligned}|g_\alpha^n(t)-g_\alpha^n(s)| &\le C_1(\alpha,T)\int\limits_s^t||\nabla\overset{n}{\boldsymbol{w}}||\mathrm{d}t+C_2(\alpha)(t-s)\le\\ &\le C_1(\alpha,T)(t-s)^{1/2}\left(\int\limits_s^t||\nabla\overset{n}{\boldsymbol{w}}||^2\mathrm{d}t\right)^{1/2}+C_2(\alpha)(t-s).\end{aligned}\tag{33}$$

On the base of the inequalities (30), (31) and(33) we can asert [20] that there exists a function $\boldsymbol{w}\in L^2(0,T);\boldsymbol{H}^1(\Omega))\cap L^\infty((0,T);\boldsymbol{H}(\Omega))$ and a subsequence $\overset{n_k}{\boldsymbol{w}}$, such that

$$\begin{aligned}\overset{n_k}{\boldsymbol{w}}(t) &\overset{\text{weakly}}{\longrightarrow} \boldsymbol{w}(t) && \text{in } \boldsymbol{L}^2(\Omega) \text{ uniformly for } t\in[0,T],\\ \overset{n_k}{\boldsymbol{w}} &\overset{\text{strong}}{\longrightarrow} \boldsymbol{w} && \text{in } L^2(0,T);\boldsymbol{H}(\Omega)),\\ \nabla\overset{n_k}{\boldsymbol{w}} &\overset{\text{weakly}}{\longrightarrow} \nabla\boldsymbol{w} && \text{in } L^2(0,T);\boldsymbol{H}(\Omega)).\end{aligned}$$

Moreover, $\boldsymbol{w}$ depends continuously on t in the weak topology of $\boldsymbol{L}^2(\Omega)$.

The function $\boldsymbol{u}=\boldsymbol{w}+\boldsymbol{v}$ is the weak solution of the IBV(1,2,3) problem.

Firstly, we show that $\boldsymbol{w}(t)$ is weakly differentiable and

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t}\in L^2(0,T);\boldsymbol{H}^{-1}(\Omega)).$$

To prove that we integrate the relations (23) with respect to t and write

$$-\int\limits_0^t\left(\overset{n}{\boldsymbol{u}}\otimes\overset{n}{\boldsymbol{u}},\nabla\boldsymbol{\psi}_\alpha\right)\mathrm{d}t+\int\limits_0^t\left(\sigma(\overset{n}{\boldsymbol{u}}),\widetilde{\partial\psi_\alpha}\right)\mathrm{d}t=\int\limits_0^t(\boldsymbol{f},\boldsymbol{\psi}_\alpha)\,\mathrm{d}t-\left(\overset{n}{\boldsymbol{w}}(t),\boldsymbol{\psi}_\alpha\right)+(w_0,\boldsymbol{\psi}_\alpha)$$

where $\overset{n}{\boldsymbol{u}}=\overset{n}{\boldsymbol{w}}+\boldsymbol{v}$. We have

$$\left|\int_0^t \left(\overset{n}{\boldsymbol{u}}\otimes\overset{n}{\boldsymbol{u}},\nabla\boldsymbol{\psi}_\alpha\right)-(\boldsymbol{u}\otimes\boldsymbol{u},\nabla\boldsymbol{\psi}_\alpha)\,\mathrm{d}t\right|\leq$$

$$\leq\left|\int_0^t \left(\overset{n}{\boldsymbol{w}}-\boldsymbol{w},\overset{n}{\boldsymbol{u}}\cdot\nabla\boldsymbol{\psi}_\alpha\right)\mathrm{d}t\right|+\left|\int_0^t \left(\overset{n}{\boldsymbol{w}}-\boldsymbol{w},\nabla\boldsymbol{\psi}_\alpha\cdot\boldsymbol{u}\right)\mathrm{d}t\right|.\leq$$

By using the strong convergence of the $\overset{n}{\boldsymbol{w}}$ to $\boldsymbol{w}$ in $L^2(0,T);\boldsymbol{H}(\Omega))$, we have

$$\left|\int_0^t \left(\overset{n}{\boldsymbol{w}}-\boldsymbol{w},\overset{n}{\boldsymbol{u}}\cdot\nabla\boldsymbol{\psi}_\alpha\right)\mathrm{d}t\right|\leq C_\alpha\left(\int_0^t||\overset{n}{\boldsymbol{w}}-\boldsymbol{w}||^2\mathrm{d}t\right)^{1/2}\left(\int_0^t||\overset{n}{\boldsymbol{u}}||^2\mathrm{d}t\right)^{1/2}\overset{n\to\infty}{\longrightarrow}0,$$

by using the weak convergence $\overset{n}{\boldsymbol{w}}(t)$ to $\boldsymbol{w}(t)$ in $\boldsymbol{L}^2(\Omega)$ uniformly for $t\in[0,T)$ we have

$$\left|\int_0^t \left(\overset{n}{\boldsymbol{w}}-\boldsymbol{w},\nabla\boldsymbol{\psi}_\alpha\cdot\boldsymbol{u}\right)\mathrm{d}t\right|\overset{n\to\infty}{\longrightarrow}0.$$

Hence,

$$\lim_{n\to\infty}\int_0^t \left(\overset{n}{\boldsymbol{u}}\otimes\overset{n}{\boldsymbol{u}},\nabla\boldsymbol{\psi}_\alpha\right)\mathrm{d}t-\int_0^t(\boldsymbol{u}\otimes\boldsymbol{u},\nabla\boldsymbol{\psi}_\alpha)\,\mathrm{d}t.$$

Inequality (32) implies that there exists a tensorial function $\overline{\boldsymbol{\sigma}}$ such that $\boldsymbol{\sigma}(\overset{n_k}{\boldsymbol{w}})$ converge weakly to $\overline{\boldsymbol{\sigma}}$ in $\boldsymbol{L}^2(Q_T)$.

Consequently, we can assert that for any α the limit function u satisfies

$$-\int_0^t(\boldsymbol{u}\otimes\boldsymbol{u},\nabla\boldsymbol{\psi}_\alpha)\,\mathrm{d}t+\int_0^t\left(\overline{\boldsymbol{\sigma}},\widetilde{\partial\psi_\alpha}\right)\mathrm{d}t=\int_0^t(\boldsymbol{f},\boldsymbol{\psi}_\alpha)\,\mathrm{d}t-(\boldsymbol{w}(t),\boldsymbol{\psi}_\alpha)+(w_0,\boldsymbol{\psi}_\alpha)$$

We multiply the last equality by the derivatives $\gamma'(t)$ of a function $\gamma\in C_0^1([0,T))$ and we integrate on $[0,T]$, to obtain

$$\int_0^T(\boldsymbol{u}\otimes\boldsymbol{u},\nabla\gamma(t)\boldsymbol{\psi}_\alpha)\,\mathrm{d}t-\int_0^T\left(\overline{\boldsymbol{\sigma}},\widetilde{\partial\gamma(t)\psi_\alpha}\right)\mathrm{d}t=$$

$$-\int_0^T(\boldsymbol{f},\gamma(t)\boldsymbol{\psi}_\alpha)\,\mathrm{d}t-\int_0^T(\boldsymbol{w}(t),\partial_t\gamma(t)\boldsymbol{\psi}_\alpha)-(w_0,\gamma(0)\boldsymbol{\psi}_\alpha),$$

and, by using Lemma 2.1 we have

$$-\int_0^T (\boldsymbol{w}(t), \partial_t \boldsymbol{\phi})\,\mathrm{d}t - \int_0^T (\boldsymbol{u}\otimes\boldsymbol{u}, \nabla\boldsymbol{\phi})\,\mathrm{d}t + \int_0^T \left(\overline{\boldsymbol{\sigma}}, \widetilde{\partial\phi}\right)\mathrm{d}t = \int_0^T (\boldsymbol{f}, \boldsymbol{\phi})\,\mathrm{d}t + (w_0, \boldsymbol{\phi}(0))$$

for any $\phi \in \boldsymbol{\mathcal{V}}_T$. Consider the functional

$$\mathcal{F}(\phi) = -\int_0^T (\boldsymbol{u}\otimes\boldsymbol{u}, \nabla\boldsymbol{\phi})\,\mathrm{d}t + \int_0^T \left(\overline{\boldsymbol{\sigma}}, \widetilde{\partial\phi}\right)\mathrm{d}t - \int_0^T (\boldsymbol{f}, \boldsymbol{\phi})\,\mathrm{d}t.$$

We have, for any $\phi \in \boldsymbol{\mathcal{V}}_T$

$$|\mathcal{F}(\phi)| \le \left(C_T(||\nabla\boldsymbol{w}||_{L^2(Q_T)} + ||\boldsymbol{v}||_4^2) + ||\overline{\boldsymbol{\sigma}}||_{L^2(Q_T)} + ||\boldsymbol{f}||_{L^2(Q_T)}\right) ||\nabla\boldsymbol{\phi}||_{L^2(Q_T)}.$$

The fact that $\boldsymbol{\mathcal{V}}_T$ is dense in $\boldsymbol{L}^2((0,T);\boldsymbol{H}^1(\Omega))$ implies that $\mathcal{F}$ belongs to $\left(\boldsymbol{L}^2((0,T);\boldsymbol{H}^1(\Omega))\right)^* = \boldsymbol{L}^2((0,T);\boldsymbol{H}^{-1}(\Omega))$. Hence there exists $\boldsymbol{w}_t \in \boldsymbol{L}^2((0,T);\boldsymbol{H}^{-1}(\Omega))$ such that

$$\mathcal{F}(\phi) = -\int_0^T \langle \boldsymbol{w}_t, \boldsymbol{\phi}\rangle_{\boldsymbol{H}^{-1}(\Omega)}\mathrm{d}t.$$

So we have $\boldsymbol{w}$, $\boldsymbol{w}_t$ and $\overline{\boldsymbol{\sigma}}$ which satisfy

$$\int_0^T \langle \boldsymbol{w}_t, \boldsymbol{\phi}\rangle_{\boldsymbol{H}^{-1}(\Omega)}\mathrm{d}ti - \int_0^T (\boldsymbol{u}\otimes\boldsymbol{u}, \nabla\boldsymbol{\phi})\,\mathrm{d}t + \int_0^T \left(\overline{\boldsymbol{\sigma}}, \widetilde{\partial\phi}\right)\mathrm{d}t = \int_0^T (\boldsymbol{f}, \boldsymbol{\phi})\,\mathrm{d}t, \quad (34)$$

for any $\phi \in \boldsymbol{L}^2((0,T);\boldsymbol{H}^1(\Omega))$, and

$$\int_0^T \langle \boldsymbol{w}_t, \boldsymbol{\phi}\rangle_{\boldsymbol{H}^{-1}(\Omega)}\mathrm{d}t + \int_0^T (\boldsymbol{w}(t), \partial_t\boldsymbol{\phi})\,\mathrm{d}t + (w_0, \boldsymbol{\phi}(0)) = 0, \quad (35)$$

for any $\phi \in \boldsymbol{\mathcal{V}}_T$.

The last inequality implies

$$\int_0^t \langle \boldsymbol{w}_t, \boldsymbol{\psi}\rangle_{\boldsymbol{H}^{-1}(\Omega)}\mathrm{d}t = (\boldsymbol{w}(t), \boldsymbol{\psi}) - (\boldsymbol{w}_0, \boldsymbol{\psi}) \quad (36)$$

for any $\boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega)$, which prove that $\boldsymbol{w}(t)$ is weakly differentiable and

$$\frac{\mathrm{d}\boldsymbol{w}(t)}{\mathrm{d}t} = \boldsymbol{w}_t$$

For the future applications we note the following results

$$\lim_{n\leftarrow\infty} \int_0^T \left(\partial_t \overset{n}{\boldsymbol{w}}, \psi\right) \mathrm{d}y = \int_0^T \langle \boldsymbol{w}_t, \boldsymbol{\psi} \rangle_{\boldsymbol{H}^{-1}(\Omega)} \mathrm{d}t \tag{37}$$

for any $\boldsymbol{\psi} \in L^2((0,T), \boldsymbol{H}^1(\Omega))$ and

$$\int_0^T \langle \boldsymbol{w}_t, \boldsymbol{w} \rangle_{\boldsymbol{H}^{-1}(\Omega)} \mathrm{d}t \le \varliminf_{n\to\infty} \int_0^T \left(\partial_t \overset{n}{\boldsymbol{w}}, \overset{n}{\boldsymbol{w}}\right) \mathrm{d}t. \tag{38}$$

Next we prove that

$$\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\boldsymbol{w} + \boldsymbol{v}).$$

To do that we use Minty's trick. Let $\boldsymbol{\eta}$ be a smooth function, we have

$$\left(\boldsymbol{\sigma}(\overset{n}{\boldsymbol{w}} + \boldsymbol{v}) - \boldsymbol{\sigma}(\boldsymbol{\eta} + \boldsymbol{v}), \widetilde{\partial \overset{n}{\boldsymbol{w}}} - \widetilde{\partial \boldsymbol{\eta}}\right) > 0.$$

Using the ODE (23) we can write

$$\begin{aligned}
\int_0^T \left(\boldsymbol{\sigma}(\overset{n}{\boldsymbol{w}} + \boldsymbol{v}), \widetilde{\partial \overset{n}{\boldsymbol{w}}} - \widetilde{\partial \boldsymbol{\eta}}\right) \mathrm{d}t \quad &= -\int_0^T \left(\partial_t \overset{n}{\boldsymbol{w}}, \overset{n}{\boldsymbol{w}} - \boldsymbol{\eta}\right) \mathrm{d}t \\
&+ \int_0^T \left(\overset{n}{\boldsymbol{u}} \otimes \overset{n}{\boldsymbol{u}}, \nabla \overset{n}{\boldsymbol{w}} - \nabla \boldsymbol{\eta}\right) \mathrm{d}t + \int_0^T \left(\boldsymbol{f}, \overset{n}{\boldsymbol{w}} - \boldsymbol{\eta}\right) \mathrm{d}t
\end{aligned}$$

for any $\boldsymbol{\eta}$ of the form (8).

Hence,

$$-\int_0^T \left(\partial_t \overset{n}{\boldsymbol{w}}, \overset{n}{\boldsymbol{w}} - \boldsymbol{\eta}\right) \mathrm{d}t + \int_0^T \left(\overset{n}{\boldsymbol{u}} \otimes \overset{n}{\boldsymbol{u}}, \nabla \overset{n}{\boldsymbol{w}} - \nabla \boldsymbol{\eta}\right) \mathrm{d}t + \int_0^T \left(\boldsymbol{f}, \overset{n}{\boldsymbol{w}} - \boldsymbol{\eta}\right) \mathrm{d}t$$
$$- \int_0^T \left(\boldsymbol{\sigma}(\boldsymbol{\eta} + \boldsymbol{v}), \widetilde{\partial \overset{n}{\boldsymbol{w}}} - \widetilde{\partial \boldsymbol{\eta}}\right) \mathrm{d}t > 0. \tag{39}$$

From the (34) we know that

$$\int_0^T \left(\overline{\boldsymbol{\sigma}}, \widetilde{\partial \boldsymbol{w}} - \widetilde{\partial \boldsymbol{\eta}}\right) \mathrm{d}t + \int_0^T \langle \boldsymbol{w}_t, \boldsymbol{w} - \boldsymbol{\eta} \rangle_{\boldsymbol{H}^{-1}(\Omega)} \mathrm{d}t -$$
$$- \int_0^T (\boldsymbol{u} \otimes \boldsymbol{u}, \nabla \boldsymbol{w} - \nabla \boldsymbol{\eta}) \, \mathrm{d}t - \int_0^T (\boldsymbol{f}, \boldsymbol{w} - \boldsymbol{\eta}) \, \mathrm{d}t = 0. \tag{40}$$

By summing up (39) and (40) we can write

$$\int_0^T \left(\overline{\boldsymbol{\sigma}}, \widetilde{\partial \boldsymbol{w}} - \widetilde{\partial \boldsymbol{\eta}}\right) \mathrm{d}t - \int_0^T \left(\boldsymbol{\sigma}(\boldsymbol{\eta} + \boldsymbol{v}), \widetilde{\partial \overset{n}{\boldsymbol{w}}} - \widetilde{\partial \boldsymbol{\eta}}\right) \mathrm{d}t +$$
$$+ \int_0^T \langle \boldsymbol{w}_t, \boldsymbol{w} - \boldsymbol{\eta} \rangle_{\boldsymbol{H}^{-1}(\Omega)} \mathrm{d}t - \int_0^T \left(\partial_t \overset{n}{\boldsymbol{w}}, \overset{n}{\boldsymbol{w}} - \boldsymbol{\eta}\right) \mathrm{d}t -$$
$$- \int_0^T (\boldsymbol{u} \otimes \boldsymbol{u}, \nabla \boldsymbol{w} - \nabla \boldsymbol{\eta}) \, \mathrm{d}t + \int_0^T \left(\overset{n}{\boldsymbol{u}} \otimes \overset{n}{\boldsymbol{u}}, \nabla \overset{n}{\boldsymbol{w}} - \nabla \boldsymbol{\eta}\right) \mathrm{d}t - \int_0^T \left(\boldsymbol{f}, \overset{n}{\boldsymbol{w}} - \boldsymbol{w}\right) \mathrm{d}t > 0.$$

Taking the limit as $n \to \infty$, by using (37) and (38) we obtain

$$\int_0^T \left(\overline{\boldsymbol{\sigma}} - \boldsymbol{\sigma}(\boldsymbol{\eta} + \boldsymbol{v}), \widetilde{\partial \boldsymbol{w}} - \widetilde{\partial \boldsymbol{\eta}}\right) \mathrm{d}t > 0. \tag{41}$$

The inequality holds for any $\boldsymbol{\eta}$ of the form (8). By using Lemma (3.1) the inequality rests true for any $\boldsymbol{\eta} \in L^2((0,T); \boldsymbol{H}^1(\Omega))$. By taking $\boldsymbol{\eta} = \boldsymbol{w} + \epsilon \boldsymbol{\phi}$

$$\int_0^T \left(\overline{\boldsymbol{\sigma}} - \boldsymbol{\sigma}(\boldsymbol{w} + \boldsymbol{v} + \epsilon \boldsymbol{\phi}), -\widetilde{\partial \boldsymbol{\phi}}\right) \mathrm{d}t > 0$$

that lead to

$$\int_0^T \left(\overline{\boldsymbol{\sigma}} - \boldsymbol{\sigma}(\boldsymbol{w}+\boldsymbol{v}), -\widetilde{\partial\boldsymbol{\phi}}\right) \mathrm{d}t > 0$$

for any $\boldsymbol{\phi} \in L^2((0,T);\boldsymbol{H}^1(\Omega))$. Results that $\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\boldsymbol{w}+\boldsymbol{v})$.

3 Semi-discrete Finite Volume Method

The finite volume method (FVM) is a method for approximating the solution of a partial differential equation (PDE). It basically consists in partitioning domain Ω, on which the PDE is formulated, into small polygonal domains ω_i(control volumes) on which the unknown is approximated by a constant values.

More often, to obtain an approximation scheme, the PDE is converted into an integral on the contour of each ω_i and then the integrand is approximated. In the standard cell-centered finite volume the approximation scheme of the integral on a common face of two control volumes involves only the values of the unknown on that control volumes. Such approximation schemes work very well when the space derivatives of the unknown contribute to the integrand only by its normal components.

In the case of GNS equations the nonlinear dependence of the viscosity on the strain rate make it difficult to approximate shear stress by standard cell-centered finite volume method, since the tangential derivatives on the interface of two adjacent volumes are not easy to approximate only by the values of unknown on the two volumes. Beside that, the incompressibility constrain on the velocity field and the presences of the pressure field in the GNS equations generate new difficulties in the standard cell-centered finite volumes approximation scheme.

To handle these problems we consider a class of finite-volume schemes that includes two type of meshes, $\mathcal{T} = \{\omega_{\mathcal{I}}, \boldsymbol{r}_{\mathcal{I}}\}$ the *primal mesh* and $\widetilde{\mathcal{T}} = \{\widetilde{\omega}_{\mathcal{J}}, \widetilde{\boldsymbol{r}}_{\mathcal{J}}\}$ the *dual mesh.* The space discrete form of the GNS equations are obtained from the integral form of the balance of momentum equation and mass balance equation on primal mesh and dual mesh respectively.

For any ω_i of the primal mesh $\mathcal{T}$ the integral form of the balance of momentum equation reads as,

$$\partial_t \int_{\omega_i} \boldsymbol{u}(\boldsymbol{x},t)\mathrm{d}x + \int_{\partial\omega_i} \boldsymbol{u}\boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}s + \int_{\omega_i} \nabla p \mathrm{d}x = \int_{\partial\omega_i} \boldsymbol{\sigma}\cdot\boldsymbol{n}\mathrm{d}s \qquad (42)$$

and for any $\widetilde{\omega}_\alpha$ of the dual mesh $\widetilde{\mathcal{T}}$ the integral form of mass balance equation is given by

$$\int_{\partial\widetilde{\omega}_\alpha} \boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}s = 0 \tag{43}$$

The velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ and the pressure field $p(\boldsymbol{x},t)$ are approximated by the piecewise constant functions on the primal mesh and the dual mesh respectively,

$$\boldsymbol{u}(\boldsymbol{x},t) \approx \boldsymbol{u}_i(t),\ \forall\boldsymbol{x}\in\omega_i,\ \ p(\boldsymbol{x},t)\approx p_\alpha(t),\ \forall\boldsymbol{x}\in\widetilde{\omega}_\alpha,$$

and using certain approximation schemes of the integrals as functions of the discrete variables $\{u_i(t)\}_{i\in\mathcal{I}}$, $\{p_\alpha(t)\}_{\alpha\in\mathcal{J}}$ one obtains the space discrete form. By using the notations:

$$\begin{aligned}
\boldsymbol{\mathcal{F}}_i(\boldsymbol{u}) &\approx \textstyle\int_{\partial\omega_i}\boldsymbol{u}\boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}s,\\
\mathbf{Grad}_i(p) &\approx \textstyle\int_{\omega_i}\nabla p\mathrm{d}x,\\
\boldsymbol{\mathcal{S}}_i(\boldsymbol{u}) &\approx \textstyle\int_{\partial\omega_i}\boldsymbol{\sigma}\cdot\boldsymbol{n}\mathrm{d}s,\\
\mathrm{Div}_\alpha(\boldsymbol{u}) &\approx \textstyle\int_{\partial\widetilde{\omega}_\alpha}\boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}s,
\end{aligned} \tag{44}$$

the semi-discrete form of GNS equations, continuum with respect to time variable and discrete with respect to space variable, can be write as:

$$\begin{aligned}
m_i\frac{\mathrm{d}\boldsymbol{u}_i}{\mathrm{d}t} + \boldsymbol{\mathcal{F}}_i(\{\boldsymbol{u}\}) + \mathbf{Grad}_i(\{p\}) - \boldsymbol{\mathcal{S}}_i(\{\boldsymbol{u}\}) = 0, &\quad i\in\mathcal{I}\\
\mathrm{Div}_\alpha(\{\boldsymbol{u}\}) = 0, &\quad \alpha\in\mathcal{J}
\end{aligned} \tag{45}$$

where m_i stands for the volume of the ω_i.

Now the problem is to find the functions $\{\boldsymbol{u}_i(t)\}_{i\in\mathcal{I}}, \{p_\alpha(t)\}_{\alpha\in\mathcal{J}}$ that satisfies the differential algebraic system of equations (DAE) (45) and the initial data

$$\boldsymbol{u}_i(t)|_{t=t_0} = \boldsymbol{u}_i^0, \forall i\in\mathcal{I}. \tag{46}$$

We note that the boundary conditions for the velocity field are not explicit considered, they are taken into account by the discrete convective flux $\mathcal{F}$ and discrete stress $\mathcal{S}$.

In solving the Cauchy problem (45) and (46), an essential step is to define a primal-dual mesh $\left(\mathcal{T},\widetilde{\mathcal{T}}\right)$ which allow one to calculate the velocity field independently on the pressure field. To do that we define a quadrilateral admissible primal-dual meshes (QAPD mesh) $\left(\mathcal{T},\widetilde{\mathcal{T}}\right)$ and we define the discrete gradient of the scalar functions and the discrete divergence of the vector functions such that the discrete space of the vector fields admits an orthogonal decomposition into two subspace: one of discrete divergences free vectors fields and other consisting of vectors that are the discrete gradient of the scalar fields.

In the next subsection we introduce the QAPD mesh and we prove the decomposition formula. In the subsection 2 we define the approximation of the convective flux and the approximation of the stress force.

3.1 Quadrilateral primal-dual meshes

Let Ω be a polygonal domain in $\mathbb{R}^2$. Let $\mathcal{T} = \{\omega_{\mathcal{I}}, \boldsymbol{r}_{\mathcal{I}}\}$ be a quadrilateral mesh defined as follows:

(1) ω_i is a quadrilateral, $\overline{\cup_{i\in I}\omega_i} = \overline{\Omega}$,
(2) $\forall i \neq j \in I$ and $\overline{\omega}_i \cap \overline{\omega}_j \neq \Phi$, either $\mathcal{H}_{n-1}(\overline{\omega}_i \cap \overline{\omega}_j) = 0$ or
$\sigma_{ij} := \overline{\omega}_i \cap \overline{\omega}_j$ is a common $(n-1)-$ face of ω_i and ω_j,
(3) $\boldsymbol{r}_i \in \omega_i$, if $\omega_i = [ABCD]$, then $\boldsymbol{r}_i = [M_{AB}M_{DC}] \cap [M_{AD}M_{BC}]$
(4) for any vertex $\mathrm{P} \in \Omega$ there exists only four quadrilateral ω with the common vertex P

where M_{AB} denotes the midpoint of the line segment $[AB]$.

Let $\widetilde{\mathcal{T}} = \{\widetilde{\omega}_{\mathcal{J}}, \widetilde{\boldsymbol{r}}_{\mathcal{J}}\}$ be another mesh defined as follows:

(1) $\forall\alpha \in \mathcal{J}$, $\widetilde{\boldsymbol{r}}_\alpha$is a vertex of $\mathcal{T}$,
(2) $\widetilde{\boldsymbol{r}}_\alpha \in \widetilde{\omega}_\alpha, \forall\alpha \in \mathcal{J}$,
(3) $\forall\widetilde{\boldsymbol{r}}_\alpha \in \overline{\Omega}$, the poligon $\widetilde{\omega}_\alpha$ has the vertexes : the centres of the qudrileterals with the common vertex $\widetilde{\boldsymbol{r}}_\alpha$ and the midpoints of the sides emerging from$\widetilde{\boldsymbol{r}}_\alpha$

where by "center" of the quadrilateral we understand the intersection of the two segments determined by the midpoints of two opposed sides.

We call $(\mathcal{T},\widetilde{\mathcal{T}})$ an admissible *primal-dual* quadrilateral meshes (QAPDmesh).

We denote by $H_{\widetilde{\mathcal{T}}}(\Omega)$ the space of piecewise constant scalar functions which are constant on each volume $\widetilde{\omega}_\alpha \in \widetilde{\omega}_{\mathcal{J}}$, by $\boldsymbol{H}_{\mathcal{T}}(\Omega)$ the space of piecewise constant vectorial functions which are constant on each volume $\omega_i \in \omega_{\mathcal{I}}$ and by $\boldsymbol{H} \otimes \boldsymbol{H}_{\widetilde{\mathcal{T}}}(\Omega)$ the space of piecewise constant tensorial functions of the order two which are constant on each volume $\widetilde{\omega}_a \in \widetilde{\omega}_{\mathcal{J}}$.

For any quantity ψ which is piecewise constant on $\widetilde{\omega}_{\mathcal{J}}$ we denote by ψ_α the constant value of ψ on $\widetilde{\omega}_a$, analogous ψ_i stands for the constant value of a piecewise constant quantity ψ on $\omega_{\mathcal{I}}$.

The discrete differential operator are defined by duality, for example if a field is piecewise constant on $\widetilde{\omega}_{\mathcal{J}}$ then its discrete derivatives are piecewise

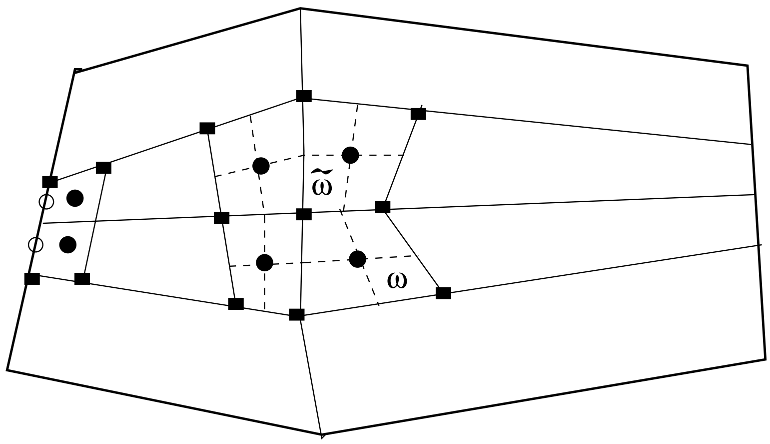


Figure 1: Quadrilateral mesh.

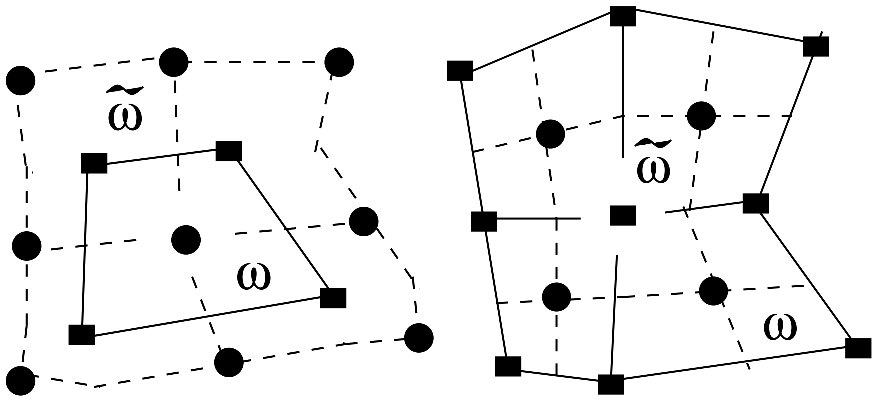


Figure 2: Dual and primal volumes

constant on $\omega_{\mathcal{I}}$.

Discrete divergence. We define a discrete divergence operator $\mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})} : \boldsymbol{H}_{\mathcal{T}}(\Omega) \to H_{\widetilde{\mathcal{T}}}(\Omega)$, by

$$\mathrm{Div}_\alpha(\boldsymbol{u}) := \int_{\partial\widetilde{\omega}_\alpha} \boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}s = \sum_i u_i^a \int_{\partial\widetilde{\omega}_\alpha\cap\omega_i} n_a \mathrm{d}s. \qquad (47)$$

Discrete gradient of a vector. A discrete gradient operator of a vector

$\boldsymbol{\partial}_{(\mathcal{T},\widetilde{\mathcal{T}})} : \boldsymbol{H}_{\mathcal{T}}(\Omega) \to \boldsymbol{H} \otimes \boldsymbol{H}_{\widetilde{\mathcal{T}}}(\Omega)$ is defined by

$$\partial_a u^b\Big|_\alpha =: \frac{1}{\mathrm{m}(\widetilde{\omega}_\alpha)} \int\limits_{\partial\widetilde{\omega}_\alpha} u^b n_a \mathrm{d}s = \frac{1}{\mathrm{m}(\widetilde{\omega}_a)} \sum_i u_i^b \int\limits_{\partial\widetilde{\omega}_\alpha \cap \omega_i} n_a \mathrm{d}s. \tag{48}$$

Discrete gradient of a scalar. A discrete gradient operator of a scalar $\mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})} : H_{\widetilde{\mathcal{T}}}(\Omega) \to \boldsymbol{H}_{\mathcal{T}}(\Omega)$ is defined by

$$\mathbf{Grad}_i(\phi) := \int\limits_{\partial\omega_i} \phi\, \boldsymbol{n} \mathrm{d}s = \sum_\alpha \phi_\alpha \int\limits_{\partial\omega_i \cap \widetilde{\omega}_\alpha} \boldsymbol{n} \mathrm{d}s. \tag{49}$$

Discrete "rotation". The "rotation" of a continous scalar field is a vector field and any vector field that result as the rotation of a scalar is divergent free vector field. The discrete counterpart of that results can be obtained by a proper definition of a discrete rotation. We define $\mathbf{rot}_{(\mathcal{T},\widetilde{\mathcal{T}})} : H_{\widetilde{\mathcal{T}}}(\Omega) \longrightarrow \boldsymbol{H}_{\mathcal{T}}(\Omega)$ by

$$\mathbf{rot}_i(\phi) := \frac{1}{\mathrm{m}(\omega_i)} \int\limits_{\partial\omega_i} \phi \mathrm{d}\boldsymbol{r} = \frac{1}{\mathrm{m}(\omega_i)} \sum_\alpha \phi_\alpha \int\limits_{\partial\omega_i \cap \widetilde{\omega}_\alpha} \mathrm{d}\boldsymbol{r}. \tag{50}$$

On the space $\boldsymbol{H}_{\mathcal{T}}(\Omega)$ we define the scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ by

$$\langle\langle \boldsymbol{u}, \boldsymbol{v} \rangle\rangle = \sum_{i \in \mathcal{I}} \boldsymbol{u}_i \cdot \boldsymbol{v}_i \tag{51}$$

and on the space $H_{\widetilde{\mathcal{T}}}(\Omega)$ we define the scalar product $\langle \cdot, \cdot \rangle$ by

$$\langle \phi, \psi \rangle = \sum_{\alpha \in \mathcal{J}} \phi_\alpha \psi_\alpha \tag{52}$$

In the next lemma we prove certain properties of the discrete derivative operators.

Lemma 3.1 *Let* $\left(\mathcal{T}, \widetilde{\mathcal{T}}\right)$ *be a primal-dual structured mesh,* $\boldsymbol{H}_{\mathcal{T}}(\Omega)$ *and* $H_{\widetilde{\mathcal{T}}}(\Omega)$ *the space of the discrete vector fields and the space of discrete scalar fields associate to it.*

Let discrete divergent be defined by (47)*, the discrete gradient be defined by* (49)*, the discrete rotation defined by* (50)*. Then:*
(a1) *Discrete Stokes formula. For any* $\boldsymbol{u} \in \boldsymbol{H}_{\mathcal{T}}(\Omega)$ *and any* $\phi \in H_{\widetilde{\mathcal{T}}}(\Omega)$ *there exists a discrete integration by parts formula*

$$\left\langle \mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\boldsymbol{u}), \phi \right\rangle + \langle\langle \boldsymbol{u}, \mathbf{Grad}(\phi) \rangle\rangle = 0. \tag{53}$$

(a2) *For any* $\psi \in H_{\widetilde{\mathcal{T}}}(\Omega)$, $\psi|_{\partial\Omega} = 0$ *one has*

$$\mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}\,\mathbf{rot}_{(\mathcal{T},\widetilde{\mathcal{T}})}\,\psi = 0. \tag{54}$$

Proof. To prove (a1) we use the fact that for any domain ω

$$\int_{\partial\omega} \boldsymbol{n}\mathrm{d}s = 0.$$

Using this result and the definitions of the two operators we have

$$\left\langle \mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\boldsymbol{u}), \phi \right\rangle = \sum_{\alpha} \mathrm{Div}_{\alpha}(\boldsymbol{u})\phi_{\alpha} = \sum_{\alpha}\left(\sum_{i} u_i^a \int\limits_{\partial\widetilde{\omega}_{\alpha}\cap\omega_i} n_a \mathrm{d}s\right)\phi_{\alpha} =$$

$$= \sum_{i} u_i^a \left(\sum_{\alpha}\phi_{\alpha}\int\limits_{\partial\widetilde{\omega}_{\alpha}\cap\omega_i} n_a\mathrm{d}s\right) = -\sum_{i} u_i^a\left(\sum_{\alpha}\phi_{\alpha}\int\limits_{\widetilde{\omega}_{\alpha}\cap\partial\omega_i} n_a\mathrm{d}s\right) =$$

$$= -\sum_{i} \boldsymbol{u}_i \mathbf{Grad}_i(\phi) = -\left\langle\left\langle \boldsymbol{u}, \mathbf{Grad}(\phi)\right\rangle\right\rangle$$

To prove (a2), using the definitions of the two operators we have

$$\mathrm{Div}_{\alpha}(\mathbf{rot}_{(\boldsymbol{\mathcal{T}},\widetilde{\boldsymbol{\mathcal{T}}})}\boldsymbol{\psi}) = \sum_{i} \mathbf{rot}_i(\psi)\cdot\int_{\partial\widetilde{\omega}_{\alpha}\cap\omega_i}\boldsymbol{n}\mathrm{d}s =$$

$$\sum_{i}\frac{1}{\mathrm{m}(\omega_i)}\sum_{\beta}\psi_{\beta}\int_{\widetilde{\omega}_{\beta}\cap\partial\omega_i}\mathrm{d}\boldsymbol{r}\cdot\int_{\partial\widetilde{\omega}_{\alpha}\cap\omega_i}\boldsymbol{n}\mathrm{d}s$$

Let $\omega_{i_a^\alpha}$, $a = \overline{1,4}$ be the primal volumes with the common vertex P_α and numbered such that $\omega_{i_a^\alpha}$ and $\omega_{i_{a+1}^\alpha}$ have a common side. For each i_a^α let $P_{\alpha_b^{i_a^\alpha}}$, $b = \overline{1,4}$ be the vertexes of the quadrilateral $\omega_{i_a^\alpha}$ anticlockwise numbered and $P_{\alpha_1^{i_a^\alpha}} = P_\alpha$. We have

$$\int_{\partial\widetilde{\omega}_{\alpha}\cap\omega_{i_a^\alpha}}\boldsymbol{n}\mathrm{d}s = \overrightarrow{\tau}_{1,3}$$

where $\overrightarrow{\tau}_{1,3}$ is a vector orthogonal to $\overrightarrow{P_{\alpha_2^{i_a^\alpha}}P_{\alpha_4^{i_a^\alpha}}}$ oriented from P_α to $P_{\alpha_3^{i_a^\alpha}}$ and $|\overrightarrow{\tau}_{1,3}| = \left|\overrightarrow{P_{\alpha_2^{i_a^\alpha}}P_{\alpha_4^{i_a^\alpha}}}\right|/2$.

$$\int_{\widetilde{\omega}_{\alpha}\cap\partial\omega_{i_a^\alpha}}\mathrm{d}\boldsymbol{r} = -\int_{\widetilde{\omega}_{\alpha_3^{i_a^\alpha}}\cap\partial\omega_{i_a^\alpha}}\mathrm{d}\boldsymbol{r} = \frac{1}{2}\overrightarrow{P_{\alpha_4^{i_a^\alpha}}P_{\alpha_2^{i_a^\alpha}}},$$

$$\int_{\widetilde{\omega}_{\alpha_2^{i_a^\alpha}} \cap \partial \omega_{i_a^\alpha}} \mathrm{d}\boldsymbol{r} = - \int_{\widetilde{\omega}_{\alpha_4^{i_a^\alpha}} \cap \partial \omega_{i_a^\alpha}} \mathrm{d}\boldsymbol{r} = \frac{1}{2} \overrightarrow{P_\alpha P_{\alpha_3^{i_a^\alpha}}}.$$

So, we have

$$\frac{1}{\mathrm{m}(\omega_{i_a})} \sum_b \psi_{\alpha_b^{i_a^\alpha}} \int_{\widetilde{\omega}_{\alpha_b^{i_a^\alpha}} \cap \partial \omega_{i_a^\alpha}} \mathrm{d}\boldsymbol{r} \cdot \int_{\partial \widetilde{\omega}_\alpha \cap \omega_{i_a}} \boldsymbol{n} \mathrm{d}s =$$

$$= \frac{1}{\mathrm{m}(\omega_{i_a})} \frac{1}{2} \overrightarrow{\tau}_{1,3} \left((\psi_\alpha - \psi_{\alpha_3^{i_a^\alpha}}) \overrightarrow{P_{\alpha_4^{i_a^\alpha}} P_{\alpha_2^{i_a^\alpha}}} + (\psi_{\alpha_2^{i_a^\alpha}} - \psi_{\alpha_4^{i_a^\alpha}}) \overrightarrow{P_\alpha P_{\alpha_3^{i_a^\alpha}}} \right) =$$

$$= \psi_{\alpha_2^{i_a^\alpha}} - \psi_{\alpha_4^{i_a^\alpha}}$$

Finally by summing up for $a = \overline{1,4}$ we have

$$\mathrm{Div}_\alpha(\mathbf{rot}_{(\boldsymbol{\mathcal{T}}, \widetilde{\boldsymbol{\mathcal{T}}})} \boldsymbol{\psi}) = \sum_a \frac{1}{\mathrm{m}(\omega_{i_a})} \sum_b \psi_{\alpha_b^{i_a^\alpha}} \int_{\widetilde{\omega}_{\alpha_b^{i_a^\alpha}} \cap \partial \omega_{i_a^\alpha}} \mathrm{d}\boldsymbol{r} \cdot \int_{\partial \widetilde{\omega}_\alpha \cap \omega_{i_a}} \boldsymbol{n} \mathrm{d}s =$$

$$= \sum_a (\psi_{\alpha_2^{i_a^\alpha}} - \psi_{\alpha_4^{i_a^\alpha}}) = 0$$

for any α such that $P_\alpha \in \Omega$. If for some α $P_\alpha \in \partial\Omega$, we use the fact that $\psi_\beta = 0$ on any boundary dual-volumes $\widetilde{\omega}_\beta$.

Now we prove an orthogonal decomposition of the space $\boldsymbol{H}_{\mathcal{T}}(\Omega)$ that resembles the continuum case. Let $\{\Psi^\alpha\}_{\alpha \in \mathcal{J}}, \Psi^\alpha \in H_{\widetilde{\mathcal{T}}}(\Omega)$ be a base of the space $H_{\widetilde{\mathcal{T}}}(\Omega)$ given by

$$\Psi^\alpha(x) = \begin{cases} 1, & \text{if } x \in \widetilde{w}^\alpha, \\ 0, & \text{if } , x \notin \widetilde{w}^\alpha. \end{cases} \tag{55}$$

Define the discrete vector field $\boldsymbol{\mathcal{U}}^\alpha \in \boldsymbol{H}_{\mathcal{T}}(\Omega)$ by

$$\boldsymbol{\mathcal{U}}^\alpha = \mathbf{rot}(\Psi^\alpha) \tag{56}$$

Using the same convention as in proving lemma **??** we can show that the only non-vanishing values of $\boldsymbol{\mathcal{U}}^\alpha$ are given by

$$\boldsymbol{\mathcal{U}}^\alpha_{i_a^\alpha} = \frac{1}{\mathrm{m}(\omega_{i_a^\alpha})} \overrightarrow{P_{\alpha_4^{i_a^\alpha}} P_{\alpha_2^{i_a^\alpha}}}$$

Let $\boldsymbol{W}_{\mathcal{T}}(\Omega)$ be the linear closer of the set $\{\boldsymbol{\mathcal{U}}^\alpha; \alpha \in \mathrm{Int}(\mathcal{J})\}$ in the space $\boldsymbol{H}_{\mathcal{T}}(\Omega)$ and let $\boldsymbol{G}_{\mathcal{T}}(\Omega)$ be the subspace orthogonal to it, so that

$$\boldsymbol{H}_{\mathcal{T}}(\Omega) = \boldsymbol{W}_{\mathcal{T}}(\Omega) \oplus \boldsymbol{G}_{\mathcal{T}}(\Omega) \tag{57}$$

We now prove the following proposition:

Proposition 3.1 *$\boldsymbol{G}_{\mathcal{T}}(\Omega)$ consists of elements $\mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})}\phi$ with $\phi \in H_{\widetilde{\mathcal{T}}}(\Omega)$.*

Proof. Let $\boldsymbol{u} \in \boldsymbol{G}_{\mathcal{T}}(\Omega)$, *i.e*

$$\langle\langle \boldsymbol{u}, \boldsymbol{\mathcal{U}}^{\alpha} \rangle\rangle = 0,\ \forall \alpha \in \mathrm{Int}(\mathcal{J}) \tag{58}$$

We will build up a function $\phi \in H_{\widetilde{\mathcal{T}}}(\Omega)$ such that

$$\mathbf{Grad}_i(\phi) = \boldsymbol{u}_i,\ \forall i \in \mathcal{I}$$

For a given ω_i we denote by $P_{\alpha_b^i}$, $b = \overline{1,4}$ its vertexes counterclockwise numbered. The gradient of a scalar field ϕ can be written as

$$\mathbf{Grad}_i(\phi) = \overrightarrow{\tau}_{1,3}(\phi_{\alpha_3^i} - \phi_{\alpha_1^i}) + \overrightarrow{\tau}_{2,4}(\phi_{\alpha_4^i} - \phi_{\alpha_2^i})$$

where $\overrightarrow{\tau}_{1,3}$ is a vector orthogonal to $\overrightarrow{P_{\alpha_2^i}P_{\alpha_4^i}}$ oriented from $P_{\alpha_1^i}$ to $P_{\alpha_3^i}$ and $|\overrightarrow{\tau}_{1,3}| = \left|\overrightarrow{P_{\alpha_2^i}P_{\alpha_4^i}}\right|/2$ and $\overrightarrow{\tau}_{2,4}$ is a vector orthogonal to $\overrightarrow{P_{\alpha_1^i}P_{\alpha_3^i}}$ oriented from $P_{\alpha_2^i}$ to $P_{\alpha_4^i}$ and $|\overrightarrow{\tau}_{2,4}| = \left|\overrightarrow{P_{\alpha_1^i}P_{\alpha_3^i}}\right|/2$. So, we have

$$\begin{aligned} \frac{\boldsymbol{u}_i \cdot \overrightarrow{P_{\alpha_2^i}P_{\alpha_4^i}}}{\mathrm{m}(\omega_i)} &= \phi_{\alpha_4^i} - \phi_{\alpha_2^i} \\ \frac{\boldsymbol{u}_i \cdot \overrightarrow{P_{\alpha_1^i}P_{\alpha_3^i}}}{\mathrm{m}(\omega_i)} &= \phi_{\alpha_3^i} - \phi_{\alpha_1^i} \end{aligned} \tag{59}$$

Apparently one can solve the equations (59) inductively, starting from two adjacent values and following some path we can calculate the values of the scalar field ϕ on entire domain. This is not solution of the problem because, for arbitrary vector field $\boldsymbol{u}$ different paths led to different values. Now we will show that if vector field $\boldsymbol{u}$ satisfies the conditions (58) then the method of "path integration" give a solution of the problem (59).

Let $\alpha \in \mathcal{J}$ be given and let $\omega_{i_a^\alpha}$, $a = \overline{1,4}$ be the primal volumes with the common vertex P_α and numbered such that $\omega_{i_a^\alpha}$ and $\omega_{i_{a+1}^\alpha}$ have a common side. Let $P_{\alpha_b^{i_a^\alpha}}$, $b = \overline{1,4}$ be the vertexes of the quadrilateral $\omega_{i_a^\alpha}$ anticlockwise numbered and $P_{\alpha_1^{i_a^\alpha}} = P_\alpha$.

$$\begin{aligned} \frac{\boldsymbol{u}_{i_1^\alpha} \cdot \overrightarrow{P_{\alpha_4^{i_1^\alpha}}P_{\alpha_2^{i_1^\alpha}}}}{\mathrm{m}(\omega_{i_1^\alpha})} &= \phi_{\alpha_2^{i_1^\alpha}} - \phi_{\alpha_4^{i_1^\alpha}} \\ \frac{\boldsymbol{u}_{i_4^\alpha} \cdot \overrightarrow{P_{\alpha_4^{i_4^\alpha}}P_{\alpha_2^{i_4^\alpha}}}}{\mathrm{m}(\omega_{i_4^\alpha})} &= \phi_{\alpha_2^{i_4^\alpha}} - \phi_{\alpha_4^{i_4^\alpha}} \end{aligned} \tag{60}$$

$$
\begin{aligned}
\frac{\boldsymbol{u}_{i_2^\alpha} \cdot \overrightarrow{P_{\alpha_2^{i_2^\alpha}} P_{\alpha_4^{i_2^\alpha}}}}{\mathrm{m}(\omega_{i_a^\alpha})} &= \phi_{\alpha_4^{i_2^\alpha}} - \phi_{\alpha_2^{i_2^\alpha}} \\
\frac{\boldsymbol{u}_{i_3^\alpha} \cdot \overrightarrow{P_{\alpha_2^{i_3^\alpha}} P_{\alpha_4^{i_3^\alpha}}}}{\mathrm{m}(\omega_{i_a^\alpha})} &= \phi_{\alpha_4^{i_3^\alpha}} - \phi_{\alpha_2^{i_3^\alpha}}
\end{aligned}
\tag{61}
$$

From the convention of the orientation we have

$$
\phi_{\alpha_4^{i_1^\alpha}} = \phi_{\alpha_2^{i_2^\alpha}} := \phi_\beta, \phi_{\alpha_2^{i_4^\alpha}} = \phi_{\alpha_4^{i_3^\alpha}} := \phi_\gamma
$$

and

$$
\phi_{\alpha_2^{i_1^\alpha}} = \phi_{\alpha_4^{i_4^\alpha}}, \phi_{\alpha_4^{i_2^\alpha}} = \phi_{\alpha_2^{i_3^\alpha}}
$$

If one follows the "path" (60) then it obtains

$$
\phi_\gamma^{(1)} = \phi_\beta + \frac{\boldsymbol{u}_{i_1^\alpha} \cdot \overrightarrow{P_{\alpha_4^{i_1^\alpha}} P_{\alpha_2^{i_1^\alpha}}}}{\mathrm{m}(\omega_{i_1^\alpha})} + \frac{\boldsymbol{u}_{i_4^\alpha} \cdot \overrightarrow{P_{\alpha_4^{i_4^\alpha}} P_{\alpha_2^{i_4^\alpha}}}}{\mathrm{m}(\omega_{i_4^\alpha})}
$$

while if one follows the "path" (60) it obtains

$$
\phi_\gamma^{(2)} = \phi_\beta + \frac{\boldsymbol{u}_{i_2^\alpha} \cdot \overrightarrow{P_{\alpha_2^{i_2^\alpha}} P_{\alpha_4^{i_2^\alpha}}}}{\mathrm{m}(\omega_{i_a^\alpha})} + \frac{\boldsymbol{u}_{i_3^\alpha} \cdot \overrightarrow{P_{\alpha_2^{i_3^\alpha}} P_{\alpha_4^{i_3^\alpha}}}}{\mathrm{m}(\omega_{i_a^\alpha})}
$$

One obtains the same values if

$$
\sum_a \frac{\boldsymbol{u}_{i_a^\alpha} \cdot \overrightarrow{P_{\alpha_2^{i_a^\alpha}} P_{\alpha_4^{i_a^\alpha}}}}{\mathrm{m}(\omega_{i_a^\alpha})} = 0
$$

that is (58) for some α. There is a corollary of the decomposition formula.

Corollary 3.1 (Discrete Hodge formula) *Let $(\mathcal{T}, \widetilde{\mathcal{T}})$ be a QAPDmeshes. Then for any $\boldsymbol{w} \in \boldsymbol{H}_{\mathcal{T}}(\Omega)$ there exists an element $\boldsymbol{u} \in \boldsymbol{H}_{\mathcal{T}}(\Omega)$ and a scalar function $\phi \in H_{\widetilde{\mathcal{T}}}(\Omega)$ such that*

$$
\boldsymbol{w} = \boldsymbol{u} + \mathbf{Grad}(\phi), \quad \text{with } \mathrm{Div}_{(\mathcal{T},\widetilde{\mathcal{T}})}(\boldsymbol{u}) = 0. \tag{62}
$$

Proof. We search for a free divergent vector $\boldsymbol{u}$ as a linear combination

$$
\boldsymbol{u} = \sum_{a \in \mathcal{J}} \alpha_a \boldsymbol{\mathcal{U}}^a \tag{63}
$$

Inserting in the decomposition, taking scalar product with the base elements and using discrete Stokes formula one obtains a linear algebraic system of equation for the determination of the unknowns $\{\alpha_a\}_{a\in\mathcal{J}}$,

$$\left\langle\left\langle \boldsymbol{w}, \boldsymbol{\mathcal{U}}^b \right\rangle\right\rangle = \sum_{a\in\mathcal{J}} \alpha_a \left\langle\left\langle \boldsymbol{\mathcal{U}}^a, \boldsymbol{\mathcal{U}}^b \right\rangle\right\rangle \tag{64}$$

This matrix of the system is the Gram matrix of a linear independent family, hence there exist a unique solution $\boldsymbol{u}$.

Since $\langle\langle \boldsymbol{w} - \boldsymbol{u}, \mathcal{U}^a \rangle\rangle = 0$ for any base function results that $\boldsymbol{w} - \boldsymbol{u}$ is orthogonal on $\boldsymbol{G}^{\perp}$ which implies that $\boldsymbol{w} - \boldsymbol{u} \in \boldsymbol{G}$. Hence there exists $\phi \in H_{\widetilde{\mathcal{T}}}(\Omega)$ such that

$$\boldsymbol{w} - \boldsymbol{u} = \mathbf{Grad}(\phi) \tag{65}$$

3.2 Discrete convective flux and discrete stress flux

To cope with the boundary value problems one define a partition $\{\partial_k\omega\}_{k\in\mathcal{K}}$ of the boundary $\partial\Omega$ mesh induced by the primal mes i.e

$$\partial_k\omega = \partial\Omega \cup \partial\omega_{i_k}, \partial\Omega = \cup_{k\in\mathcal{K}}\partial_k\omega$$

On each $\partial_k\omega$ one approximates by a constant values $\boldsymbol{u}_{D\,k}$ the boundary data $\boldsymbol{u}_D$.

There exists several formulas to calculate the numerical convective flux (NCF), most of them are derived from the theory of hyperbolic equations. In the case of hyperbolic equation the numerical convective flux, beside the accuracy of the approximation, it must satisfy a number of conditions in order that the implied solution be physically relevant. In the case of Navier-Stokes equation at high Reynolds number the way in which NCF is evaluated is also very important. Here we revised two frequently used formulas and we propose a new formula which proved very good results in our numerical simulation.

Let us consider two adjacent volumes $\overline{\omega}_i$ and $\overline{\omega}_j$ and there common face $\sigma^{(i,j)} = \overline{\omega}_i \cap \overline{\omega}_j$.

One way to define the NCF is to approximate the integral

$$I_{i,j} = \int_{\sigma^{(i,j)}} \boldsymbol{u}\boldsymbol{u} \cdot \boldsymbol{n} \mathrm{d}s$$

by a constant values $\boldsymbol{f}^{(i,j)}(\{\boldsymbol{u}\}) \approx I_{i,j}$ and then define the $\boldsymbol{f}^{(i,j)}$ as function on $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$. The NCF $\mathcal{F}_i(\{\boldsymbol{u}\})$ in 44 is given by

$$\boldsymbol{\mathcal{F}}_i(\{\boldsymbol{u}\}) = \sum_j \boldsymbol{f}^{(i,j)}(\{\boldsymbol{u}\})$$

where the summation index j run for all ω_j that share a common face with ω_i.

The upstream NCF is given by

$$\boldsymbol{f}^{(i,j)}(\{\boldsymbol{u}\}) = \mathrm{m}(\sigma^{(i,j)})\boldsymbol{f}^{(i,j)}_{\text{ups}}(\boldsymbol{u}_i, \boldsymbol{u}_j) \tag{66}$$

where

$$\begin{aligned} \boldsymbol{f}^{(i,j)}_{\text{ups}}(\boldsymbol{u}, \boldsymbol{v}) &= w^+\boldsymbol{u} - w^-\boldsymbol{v} \\ w &= 1/2\,(\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{n}^{(i,j)} \end{aligned} \tag{67}$$

Another possible choice is so-called skew-symmetric

$$\begin{aligned} \boldsymbol{f}^{(i,j)}_{\text{sks}}(\boldsymbol{u}, \boldsymbol{v}) &= w\frac{\boldsymbol{u}+\boldsymbol{v}}{2} \\ w &= \frac{(\boldsymbol{u}+\boldsymbol{v})}{2}\cdot\boldsymbol{n}^{(i,j)} \end{aligned} \tag{68}$$

It is ready to verifies that the both $\boldsymbol{f}^{(i,j)}_{\text{sks}}$ and $\boldsymbol{f}^{(i,j)}_{\text{ups}}$ verifies the consistency and conservatively properties,

$$\begin{aligned} \boldsymbol{f}^{(i,j)}(\boldsymbol{u}, \boldsymbol{u}) &= \boldsymbol{u}\boldsymbol{u}\cdot\boldsymbol{n}^{(i,j)} && \text{(consistency)} \\ \boldsymbol{f}^{(i,j)}(\boldsymbol{u}, \boldsymbol{v}) + \boldsymbol{f}^{(j,i)}(\boldsymbol{v}, \boldsymbol{u}) &= 0 && \text{(conservativity)} \end{aligned} \tag{69}$$

Another way, which we propose, to define the NCF is to consider the tensorial product $\boldsymbol{u}\oplus\boldsymbol{u}$ constant on the dual mesh and then calculate the NCF as follows. For any control volume ω_i which not lies on the boundary $\boldsymbol{\mathcal{F}}$ is given by

$$\mathcal{F}^a_i = \sum_\alpha (u^a u^b)_\alpha \int_{\widetilde{\omega}_\alpha\cap\partial\omega_i} n_b \mathrm{d}s \tag{70}$$

The tensorial product $\boldsymbol{u}\oplus\boldsymbol{u}$ is approximated by

$$(u^a u^b)_\alpha = \frac{1}{\mathrm{m}(\widetilde{\omega}_\alpha)}\int_{\widetilde{\omega}_\alpha} u^a \mathrm{d}x\,\frac{1}{\mathrm{m}(\widetilde{\omega}_\alpha)}\int_{\widetilde{\omega}_\alpha} u^b \mathrm{d}x \tag{71}$$

The numerical stress flux is set up by considering that the gradient of the velocity is piecewise constant on the dual mesh. This fact implies that the stress tensor is also piecewise constant on the dual mesh. So we can write for the numerical stress flux

$$\boldsymbol{\mathcal{S}}_i(\boldsymbol{u}) = \sum_\alpha \boldsymbol{\sigma}_\alpha(\boldsymbol{u})\cdot\int\limits_{\partial\omega_i\cap\widetilde{\omega}_\alpha} \boldsymbol{n}\mathrm{d}s \tag{72}$$

The values $\boldsymbol{\sigma}_\alpha(\boldsymbol{u})$ is evaluated as

$$\boldsymbol{\sigma}_\alpha(\boldsymbol{u}) = 2\nu(|\boldsymbol{D}_\alpha(\boldsymbol{u})|)\boldsymbol{D}_\alpha(\boldsymbol{u}) \tag{73}$$

where the discrete strain rate tensor $\boldsymbol{D}_\alpha$ is given by

$$D_{ab}(\boldsymbol{u})|_\alpha = \frac{1}{2}\left(\partial_a u^b + \partial_b u^a\right)\Big|_\alpha \tag{74}$$

Dirichlet Boundary conditions The boundary conditions of the velocity are taken into account through the numerical convective flux and numerical stress flux. If for some α the dual volume $\widetilde{\omega}_\alpha$ intersects the boundary $\partial\Omega$ the gradient of the velocity is given by:

$$\partial_a u^b\Big|_\alpha = \frac{1}{\mathrm{m}(\widetilde{\omega}_\alpha)} \int\limits_{\partial\widetilde{\omega}_\alpha} u^b n_a \mathrm{d}s =$$

$$= \frac{1}{\mathrm{m}(\widetilde{\omega}_\alpha)} \left(\int\limits_{\partial_{\mathrm{ext}}\widetilde{\omega}_\alpha} u_D^b n_a \mathrm{d}s + \sum_i u_i^b \int\limits_{\partial_{\mathrm{int}}\widetilde{\omega}_\alpha \cap \omega_i} n_a \mathrm{d}s. \right) \tag{75}$$

For a primal volume ω_i adjacent to the boundary $\partial\Omega$ the NCF (70) is given by

$$\mathcal{F}_i^a = u_D^a u_D^b \int\limits_{\partial\Omega\cap\partial\omega_i} n_b \mathrm{d}s + \sum_\alpha (u^a u^b)_\alpha \int\limits_{\Omega\cap\partial\omega_i} u n_b \mathrm{d}s. \tag{76}$$

The upstream (66) and skew-symmetric (68) flux functions is evaluated as follows. If $\boldsymbol{f}^{(i,j)}(\boldsymbol{u}_i, \boldsymbol{u}_j)$ define the the numerical convective flux trough the face $\omega_i \cap \omega_j$ then the numerical convective flux through the face $\partial_k\omega$ is defined by $\boldsymbol{f}^{(i_k)}(\boldsymbol{u}_{i_k}, \boldsymbol{u}_{D\,k})$

4 Fully-Discrete Finite Volume Method

We set up a time integration scheme of the Cauchy problem (45) and (46) that determines the velocity field independently on the pressure field. The pressure field results from the discrete balance momentum equation (45-1). The scheme look like Galerkin method and it make use of the orthogonal decomposition (57) of the space $\boldsymbol{H}_\mathcal{T}(\Omega)$ and the set of the divergence free vectorial fields $\{\boldsymbol{\mathcal{U}}^{\boldsymbol{\alpha}}\}_{\alpha\in\mathcal{J}^0}$.

We write unknown velocity field $\boldsymbol{u}(t)$ as linear combination of $\{\boldsymbol{\mathcal{U}}^{\boldsymbol{\alpha}}\}_{\alpha\in\mathcal{J}^0}$

$$\boldsymbol{u} = \sum_\alpha \xi_\alpha(t)\boldsymbol{\mathcal{U}}^\alpha \tag{77}$$

and the coefficients $\xi_\alpha(t)$ are required to satisfy the ordinary differential equations

$$\sum_\alpha \frac{\mathrm{d}\xi_\alpha}{\mathrm{d}t} \left\langle\left\langle m\boldsymbol{\mathcal{U}}^\alpha, \boldsymbol{\mathcal{U}}^\beta \right\rangle\right\rangle + \left\langle\left\langle \boldsymbol{\mathcal{F}}(\xi), \boldsymbol{\mathcal{U}}^\beta \right\rangle\right\rangle - \left\langle\left\langle \boldsymbol{\mathcal{S}}(\xi), \boldsymbol{\mathcal{U}}^\beta \right\rangle\right\rangle = 0, \ \forall \beta \in \mathcal{J}^0. \tag{78}$$

with the initial conditions

$$\sum_\alpha \xi_\alpha(0) \left\langle\left\langle \boldsymbol{\mathcal{U}}^\alpha, \boldsymbol{\mathcal{U}}^\beta \right\rangle\right\rangle = \left\langle\left\langle \boldsymbol{u}^0, \boldsymbol{\mathcal{U}}^\beta \right\rangle\right\rangle, \ \forall \beta \in \mathcal{J}^0 \tag{79}$$

If the functions ξ_α solve (78) and (79) then $m\dfrac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{\mathcal{F}}(\{\boldsymbol{u}\}) - \boldsymbol{\mathcal{S}}(\{\boldsymbol{u}\})$ belongs to the space $\boldsymbol{G}_\mathcal{T}(\Omega)$ which implies that there exists a scalar fields $p(t)$ such that

$$-\mathbf{Grad}_{(\mathcal{T},\widetilde{\mathcal{T}})} p = m\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{\mathcal{F}}(\{\boldsymbol{u}\}) - \boldsymbol{\mathcal{S}}(\{\boldsymbol{u}\}) \tag{80}$$

As concerning the initial conditions (46) we note that for $t = 0$ the solution (77) equals not u^0 but the projection of it on the space $\boldsymbol{W}_\mathcal{T}(\Omega)$.

Now we develop a time integration scheme for the equation (78) derived from two steps implicit backward differentiation formulae (BDF).

Let $\{t^n\}$ be the sequence of the moments of time we denote by $\xi_\alpha^n = \xi_\alpha(t^n), \boldsymbol{u}^n = \sum_\alpha \xi_\alpha^n \boldsymbol{\mathcal{U}}^\alpha$. Supposing that one knows the values $\{\xi^{n-1}, \xi^n\}$ one calculates the values ξ^{n+1} at the next moment of time t_{n+1} as follows. Define a polynomial $P(t)$ which interpolates the unknowns ξ^{n+1} and knows $\{\xi^{n-1}, \xi^n\}$ at the moments of time t^{n+1}, t^n, t^{n-1} respectively,

$$P_\alpha(t) = \xi_\alpha^{n+1} \frac{(t-t^n)(t-t^{n-1})}{(t^{n+1}-t^n)(t^{n+1}-t^{n-1})} + \xi_\alpha^n \frac{(t-t^{n+1})(t-t^{n-1})}{(t^n-t^{n+1})(t^n-t^{n-1})} +$$

$$+\xi_\alpha^{n-1} \frac{(t-t^{n+1})(t-t^n)}{(t^{n-1}-t^{n+1})(t^{n-1}-t^n)}$$

The unknowns ξ^{n+1} are determined by imposing to the polynomial $P(t)$ to satisfies the equations (78).

For a constant time step $\triangle t$ one has

$$\frac{\mathrm{d}P_\alpha(t^{n+1})}{\mathrm{d}t} = \left(\frac{3}{2}\xi_\alpha^{n+1} - 2\xi_\alpha^n + \frac{1}{2}\xi_\alpha^{n-1}\right) / \triangle t.$$

which led to the following nonlinear equations for ξ^{n+1}

$$\sum_I \frac{3}{2}\xi_\alpha^{n+1}\left\langle\left\langle m\boldsymbol{\mathcal{U}}^\alpha,\boldsymbol{\mathcal{U}}^\beta\right\rangle\right\rangle + \quad \triangle t\left\langle\left\langle\boldsymbol{\mathcal{F}}(\xi^{n+1}),\boldsymbol{\mathcal{U}}^\beta\right\rangle\right\rangle - \triangle t\left\langle\left\langle\boldsymbol{\mathcal{S}}(\xi^{n+1}),\boldsymbol{\mathcal{U}}^\beta\right\rangle\right\rangle = \\ = \left\langle\left\langle 2\boldsymbol{u}^n - 0.5\boldsymbol{u}^{n-1}, m\boldsymbol{\mathcal{U}}^\beta\right\rangle\right\rangle \tag{81}$$

To overcome the difficulties implied by the nonlinearity instead we consider a linear algorithm. A linear version read us:

$$\sum_\alpha \frac{3}{2}\lambda_\alpha^{n+1}\left\langle\left\langle m\boldsymbol{\mathcal{U}}^\alpha,\boldsymbol{\mathcal{U}}^\beta\right\rangle\right\rangle - \triangle t\left\langle\left\langle\boldsymbol{\mathcal{S}}(\boldsymbol{u}^n;\lambda^{n+1}),\boldsymbol{\mathcal{U}}^\beta\right\rangle\right\rangle = \\ 0.5\left\langle\left\langle\boldsymbol{u}^n - \boldsymbol{u}^{n-1}, m\boldsymbol{\mathcal{U}}^\beta\right\rangle\right\rangle - \\ -\triangle t\left\langle\left\langle\frac{3}{2}\boldsymbol{\mathcal{F}}(\boldsymbol{u}^n) - \frac{1}{2}\boldsymbol{\mathcal{F}}(\boldsymbol{u}^{n-1}),\boldsymbol{\mathcal{U}}^\beta\right\rangle\right\rangle + \triangle t\left\langle\left\langle\boldsymbol{\mathcal{S}}(\boldsymbol{u}^n),\boldsymbol{\mathcal{U}}^\beta\right\rangle\right\rangle \tag{82}$$

where

$$\lambda^{n+1} := \xi^{n+1} - \xi^n$$

For the first step one can use a Euler step

$$\sum_\alpha \lambda_\alpha^{n+1}\left\langle\left\langle m\boldsymbol{\mathcal{U}}^\alpha,\boldsymbol{\mathcal{U}}^\beta\right\rangle\right\rangle - \triangle t\left\langle\left\langle\boldsymbol{\mathcal{S}}(\boldsymbol{u}^n;\lambda^{n+1}),\boldsymbol{\mathcal{U}}^\beta\right\rangle\right\rangle = \\ -\triangle t\left\langle\left\langle\boldsymbol{\mathcal{F}}(\boldsymbol{u}^n),\boldsymbol{\mathcal{U}}^\beta\right\rangle\right\rangle + \triangle t\left\langle\left\langle\boldsymbol{\mathcal{S}}(\boldsymbol{u}^n),\boldsymbol{\mathcal{U}}^\beta\right\rangle\right\rangle \tag{83}$$

In the both (82),(83) schemes we use the notations

$$\mathcal{S}(\boldsymbol{u}^n;\lambda^{n+1}) = 2\nu(|\boldsymbol{D}(\boldsymbol{u}^n)|)\sum_\alpha \lambda_\alpha^{n+1}\boldsymbol{D}(\boldsymbol{\mathcal{U}}^{\boldsymbol{\alpha}})$$

5 Numerical Results

5.1 1D Couette flow

A very simple case to test the response of the numerical model to the numerical approximation of the viscosity is 1D flow. A nontrivial example, and rich in application, is the evolutionary Couette flow . The fluid flows through a flat channel of thickness h between two fixed horizontal plates. The length L and the width W of the channel are more longer and wider than it thick h, so we may assume fully developed flow and neglect edge effects. Take the x direction to be the direction of the flow and parallel with the plates, and y direction perpendicular to the plates.

We assume that there is no flow in the direction perpendicular to the plate and all field variables is not depending on the x variable. According

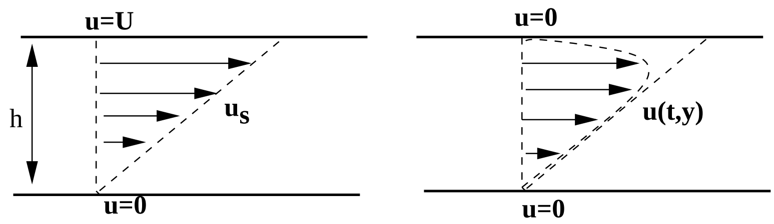


Figure 3: Couette 1D flow. Steady solution (left) and evolutionary profile of the velocity from non-slip boundary condition (right).

to the incompressibility constraint the x component of the velocity, u, is depending only the space variable y and the time variable t. The momentum balance equations read as,

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\nu\left(\left|\frac{\partial u}{\partial y}\right|\right), \frac{\partial u}{\partial y}\right) \\ 0 &= -\frac{\partial p}{\partial y}. \end{aligned} \tag{84}$$

From the last equation one can concludes that the pressure fields is a constant function across the channel.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial y}\left(\nu\left(\left|\frac{\partial u}{\partial y}\right|\right) \frac{\partial u}{\partial y}\right). \tag{85}$$

ODE Approximation

Let $0 = x_{1/2} < x_{1+1/2} < \cdots x_{N+1/2} = h$ be the knots of the partition of the interval $[0, h]$, $\omega_i = \left[x_{i-1/2}, x_{i+1/2}\right]$ be a control volume and $x_i = (x_{i-1/2} + x_{i+1/2})/2$ be the center of ω_i, $i = \overline{1, N}$.

The discrete form of the equation (85) is given by

$$m_i \frac{\mathrm{d}u_i}{\mathrm{d}t} = \nu(u_{i+1}, u_i)\frac{u_{i+1} - u_i}{d_{i+1,i}} - \nu(u_i, u_{i-1})\frac{u_i - u_{i-1}}{d_{i,i-1}}, \quad i = 1, ., ., ., N \tag{86}$$

where m_i is the length of the volume control ω_i, $d_{i,j} = |x_i - x_j|$ and $\nu(u_{i+1}, u_i)$ is a discrete approximation of the constitutive function $\nu\left(\left|\frac{\partial u}{\partial y}\right|\right)$ supposed to be a continue function. A very simple choice for numerical viscosity is

$$\nu(u_{i+1}, u_i) = \nu\left(\left|\frac{u_{i+1} - u_i}{d_{i+1,i}}\right|\right). \tag{87}$$

In the r.h.s of the ODE (86), the quantities u_0 and u_{N+1} equal the boundary data. So, according with boundary data (**??**), we set

$$u_{N+1}(t) = 0, u_0(t) = 0. \tag{88}$$

The initial conditions are given by

$$u_i(0) = u_s(x_i). \tag{89}$$

To resume, the ODE approximation of the PDE (85) with the boundary conditions (**??**) and initial condition (**??**) is given by (86), (87), (88) and (89).

There exists two important properties of the solution of the ODE model, namely the maximal principle property and the monotony of the "kinetic energy". The maximal principle is equal true for general boundary data and initial condition. Assume that there exists two constants α and β such that

$$\alpha < u_i(0), u_0, u_{N+1} < \beta, i = 1, \cdots, N$$

Proposition 5.1 *Let* $(0, T)$ *be the maximal interval of the time of the existence of the solution of the ODE approximation. Then*

$$\alpha < u_i(t) < \beta, t \in (0, T), i = 1, \cdots, N \tag{90}$$

If $u_0 = u_{N+1} = 0$ *then*

$$\sum_i m_i u_i^2(t_1) <= \sum_i m_i u_i^2(t_2), t_1 < t_2 \tag{91}$$

Proof. To prove that the solution $\{u(t)\}$ stay in the N-dimensional rectangle $\mathcal{Q} = [\alpha, \beta]^N$ we will show that any face of the rectangle is an entry face for the trajectories of the ODE. Consider a face $u_i = \alpha$ and we assume there exists a moment of time t^* such that $\{u(t)\} \in \mathcal{Q}$ for $t < t^*$ and for $t = t^*$ $u_i = \alpha$ and remainder of the components still stay in $\mathcal{Q}$. We have

$$m_i \left.\frac{\mathrm{d}u_i}{\mathrm{d}t}\right|_{t=t^*} = \nu(u_{i+1}, \alpha)\frac{u_{i+1} - \alpha}{d_{i+1,i}} - \nu(\alpha, u_{i-1})\frac{\alpha - u_{i-1}}{d_{i,i-1}} \geq 0$$

then $u_i \geq \alpha$. To prove the second affirmation we multiply each $i-$ equation (86) with u_i,

$$\sum_i m_i \frac{\mathrm{d}u_i}{\mathrm{d}t} u_i = \sum_i \left(\nu(u_{i+1}, u_i)\frac{u_{i+1} - u_i}{d_{i+1,i}} - \nu(u_i, u_{i-1})\frac{u_i - u_{i-1}}{d_{i,i-1}} \right) u_i$$

and then sum up. After some manipulation we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i m_i \frac{u_i^2}{2} + \sum_{i=0}^{N} \nu(u_{i+1}, u_i)\frac{(u_{i+1}-u_i)^2}{d_{i+1,i}} = 0$$

which yields (91).

By using a BDF time integration scheme we obtain

$$\begin{aligned} m_i\left(\frac{a}{\triangle t_{n+1}}u_i^{n+1} - w_i^P(t_{n+1})\right) = {} & \nu_{i+1/2}(u^{n+1})\frac{u_{i+1}^{n+1}-u_i^{n+1}}{d_{i+1,i}} - \\ & -\nu_{i-1/2}(u^{n+1})\frac{u_i^{n+1}-u_{i-1}^{n+1}}{d_{i,i-1}} \end{aligned} \tag{92}$$

where

$$\nu_{i+1/2}(v) = \nu(v_{i+1}, v_i),$$

a is a constant specific to the order of the method and $\triangle t_{n+1} = t_{n+1} - t_n$ is the time step. The term $w^P(t_{n+1})$ in the l.h.s of (92) is a known quantities as function of $\{u^n, ..., u^{n-k}\}$,

$$w_i^P(t_{n+1}) = -\dot{\omega}_i^P(t_{n+1}) + \frac{a}{\triangle t_{n+1}}\omega_i^P(t_{n+1}) = \sum_{j=0}^{k}\beta_j^{n+1}q_j(t_{n+1})u^{n-j}. \tag{93}$$

where

$$\beta_j^{n+1} = -\sum_{l=0, l\neq j}^{k}\frac{1}{t_{n+1}-t_{n-l}} + \frac{a}{\triangle t_{n+1}}$$

The implicit Euler method can be also described by (92) and (93). In such case the constant $a = 1$ and

$$w_i^P(t_{n+1}) = \frac{1}{\triangle t_{n+1}}u_i^n \tag{94}$$

In the case of Newtonian fluid one deals with a linear system of algebraic equation that can be ready solved. In the case of non-Newtonian fluid we are facing with a nonlinear system and we must develop a nonlinear solver. We build up an iterative algorithm to solve nonlinear equations (94) that proved good results.

Our algorithm read as

$$m_i\left(\frac{a}{\triangle t_{n+1}}u_i^{n+1,k}-w_i^P(t_{n+1})\right) = \nu_{i+1/2}(u^{n+1,k-1})\frac{u_{i+1}^{n+1,k}-u_i^{n+1,k}}{d_{i+1,i}}- -\nu_{i-1/2}(u^{n+1,k-1})\frac{u_i^{n+1,k}-u_{i-1}^{n+1,k}}{d_{i,i-1}} \quad (95)$$

As the initial guess in the algorithm we use

$$u^{n+1,0}=u^n \quad (96)$$

The iterative process is considered successful if for a given tolerance ε the residue $\mathcal{R}$

$$\mathcal{R}_i(\boldsymbol{u}) = m_i\left(\frac{a}{\triangle t_{n+1}}u_i-w_i^P(t_{n+1})\right)-\nu_{i+1/2}(u)\frac{u_{i+1}-u_i}{d_{i+1,i}}- -\nu_{i-1/2}(u)\frac{u_i-u_{i-1}}{d_{i,i-1}} \quad (97)$$

satisfies

$$\left|\left|\mathcal{R}\left(\boldsymbol{u}^{n+1,k}\right)\right|\right|_\infty \le \varepsilon \quad (98)$$

in a maximum $LMAX$ iterations.

Proposition 5.2 (a) *For any time step $\triangle t_{n+1}$ there exists a solution of the equations* (92). *The solution satisfies the inequality*

$$||u^{n+1}|| \le \frac{M}{m}\frac{\triangle t_{n+1}}{a}\,||w^P||$$

(b) *For each iteration step l there exists a unique solution $u^{n+1,l}$ of the equation* (95) *which satisfies*

$$\frac{\triangle t_{n+1}}{a}\inf_j w_j^P \le u_i^{n+1,l} \le \frac{\triangle t_{n+1}}{a}\sup_j w_j^P$$

Proof. (a) To prove the existence of a solution of the nonlinear equation (92) we use an application of Browder fixed-point point theorem that assert that if there exists a positive real constant η such that $(\mathcal{R}(u),u)\ge 0$ for any $||u||=\eta$ and the function $\mathcal{R}(u)$ is a continue function on the sphere $||u||\le\eta$ then there exist a solution of the equation

$$\mathcal{R}_i(u)=0$$

on that sphere, [26]. One has

$$(\mathcal{R}(u), u) = \frac{a}{\triangle t_{n+1}} \|u\|_2^2 - \langle u, w^P \rangle + \sum_i \nu_{i+1/2}(u) \frac{(u_{i+1} - u_i)^2}{d_{i+1,i}}$$

and from that and Cauchy-Schwartz inequality results

$$(\mathcal{R}(u), u) \geq \frac{a}{\triangle t_{n+1}} \|u\|_2^2 - \langle u, w^P \rangle \geq \|u\|_2 \left(\frac{a}{\triangle t_{n+1}} m \|u\| - M \left\| w^P \right\| \right)$$

(b) The existences and uniqueness of the solution of linear system result from the fact that the matrix of system is a nonsingular matrix. To prove the boundedness of a solution $u^{n+1,l}$ we analyse an index i_0 for which $u_{i_0}^{n+1,l} \leq u_i^{n+1,l}; \forall i$. We have

$$m_{i_0} \left(\frac{a}{\triangle t_{n+1}} u_{i_0}^{n+1,k} - w_{i_0}^P(t_{n+1}) \right) \geq 0,$$

(the r.h.s of i_0–equation in (95) is a positive number), hence

$$u_i^{n+1,l} \geq u_{i_0}^{n+1,l} \geq \frac{\triangle t_{n+1}}{a} w_{i_0}^P \geq \frac{\triangle t_{n+1}}{a} \inf_i w_i^P$$

The upper bound case can be proved in same manner.

The analytical solution of the problem (85) with the boundary conditions (**??**) and the initial data (**??**) is given by

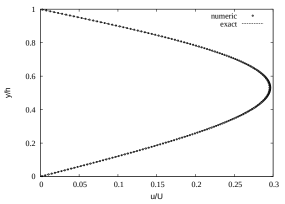
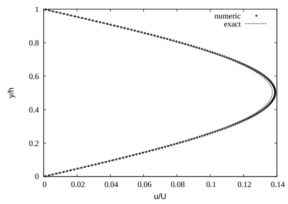
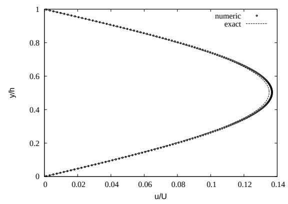
$$u_{exact}(t, y) = U \frac{2}{\pi} \sum_j \frac{1}{j} \sin \frac{j\pi}{h} (h - y) \mathrm{e}^{-\frac{j^2 \pi^2}{h^2} \nu t} \tag{99}$$

The figure (4) summarize the essential facts about our numerical code

The next aim is to test the nonlinear solver. For that we consider the Carreau-Yasuda model and we analyse the response of the code to some external data entry, like residual error ε, local time error TOL, maximal admissible time step $\triangle t_{max}$. The parameters in the constitutive model are $\nu_{inf} = 1.57 \times 10^{-6} \mathrm{m}^2\mathrm{s}^{-1}$, $\nu_0 = 15.7 \times 10^{-6} \mathrm{m}^2\mathrm{s}^{-1}$, $\Lambda = 0.11\mathrm{s}$, $n = 0.392$ and $a = 0.644$, human blood [16], [10] .

Remark 1. As we can see in the table (1) the low tolerance in local time error implies high number of iterations in the iterative process and large residual error in the solution of nonlinear equations destroy the performance of the time integration scheme.

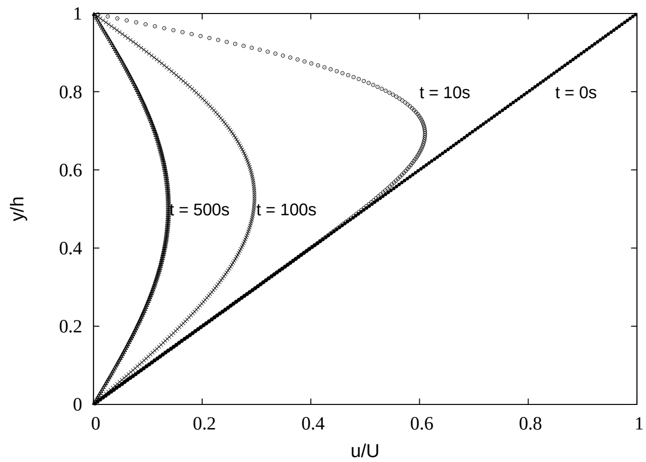
Remark 2. In the paper [9] the authors studied the problem of cessing of Cuette flow of a Bingham plastic fluid. By numerically simulation they

$t = 100$s	$t = 500$s	$t = 1000$s
$e_\infty = 0.5\%$ 	$e_\infty = 0.4\%$ 	$e_\infty = 0.4\%$ 

Comparison of analytical solution with the numerical solution

$$e_\infty = \sup_i \frac{|u_i(t) - u_{exact}(t, x_i)|}{||u_{exact}(t,x)||_\infty} \times 100$$

BDF of the fix order 4, $\triangle t_{max} = 2.$s, $TOL = 10^{-3} \div 10^{-7}$



Evolution of the velocity. Numerical solution.

Figure 4: Newtonian Fluid with $\nu = 15.7 \times 10^{-6}$. The evaluation was made on an uniform distributed net wit 200 internal knots. The distance be between walls was taken $h = 0.1$m and the relative velocity of the wall was $U = 0.4\text{ms}^{-1}$.

have showed that the velocity decay to zero in a finite time, see also the references in that paper about the existence of finite stopping time. In the case of the pseudo-plastic fluid our numerical simulation show that its velocity is greater than the velocity of a newtonian fluid, hence there is no finite stopping time.

Table 1: *Response of the code to the data entry* TOL *and* ε. $NREJECTED$ *numbers of time step rejected, local error is bigger than tolerance* TOL, $NSTEP$ *number of time step accepted local error is smaller than tolerance* TOL, $MAXITER$ *the greater number of iterations in the iterative process encountered on the time integration interval* $(0, T)$, $T = 100$s. *All calculations was made on a uniform distributed net of 200 internal knots.*

	$NREJECTED$	$NSTEP$	$MAXITER$
$\varepsilon = 10^{-5}$	374	2739	2
$\varepsilon = 10^{-7}$	190	1590	3
$\varepsilon = 10^{-9}$	8	609	5
$\varepsilon = 10^{-11}$	FAILED		
$TOL = 10^{-7}$, $\triangle t_{max} = 2.5$s			
$TOL = 10^{-3}$	0	136	7
$TOL = 10^{-5}$	0	199	6
$TOL = 10^{-7}$	8	196	5
$TOL = 10^{-8}$	FAILED		
$\varepsilon = 10^{-9}$, $\triangle t_{max} = 2.5$s			

5.2 Lid Driven Cavity Flow

The fluid is moving in a rectangular box, the side and bottom walls are static while the top wall is moving across the cavity with a constant velocity $u = U, v = 0$ see figure 6. We assume the non-slip boundary conditions on the walls. The pseudo-plastic fluid is modeled by the Carreau-Yasuda law.

In the current study the problem was solved for a series of rectangular regular or non-regular meshes. The code incorporate: the time integration scheme (82) and (83); the numerical convective flux $\mathcal{F}$ defined by the formulae (70), (71) (76) and the numerical stress flux $\mathcal{S}$ defined by the formulae(72),(73), (74), (48), (75). In the all sets of the numerical simulations we consider that at the initial time the fluid is static.

The first set of computations compare the behavior of a pseudo-plastic fluid with two viscous fluid. The figure 7 shows the contours plot of the steady solutions for the three type of the fluids. Each flow consists of a core of fluid undergoing solid body rotation and a small regions in the bottom corners of counter-rotating vortex. The intensity of the counter-rotating vortex is decreasing with respect to viscosity. The velocity profile along the vertical centerline is shown in the figure (8). We observe that in the lower part of the the cavity the fluid is moving in contrary sense to the sense of

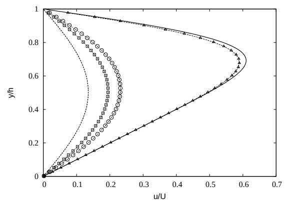
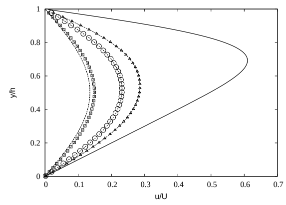
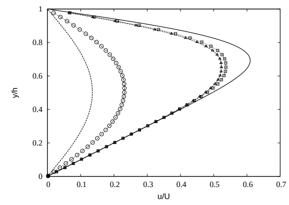
$a = 0.1, 0.644, 0.9$	$n = 0.1, 0.392, 0.9$	$\Lambda = 0.11, 10.0, 14.1$
		
The reference parameters: $a = 0.644$, $n = 0.392$, $\Lambda = 0.11\mathrm{s}$		

Figure 5: The response of the numerical model to the variation of the parameters in Carreau-Yasuda model. The line draws the profiles of the velocity of Newtonian fluids with viscosity ν_0 (left plot) and ν_∞ (right plot). The line-point draws the profiles of the velocity of pseudo-plastic fluids for several values of the parameters, the smallest value corresponds the left plot.

the motion of the top wall. The maximum of the negative velocity depends on the viscosity, like a decreasing function, of the fluid.

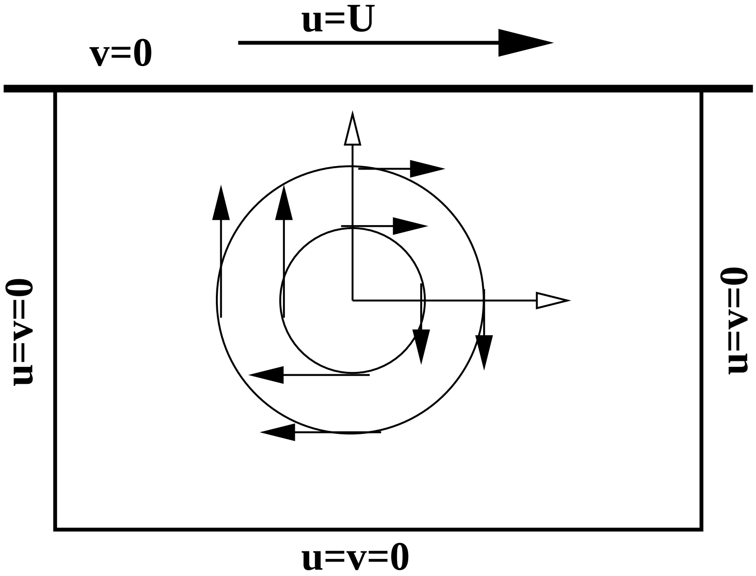


Figure 6: Lid Driven 2D Cavity Flow

The second set of computations analyse the response of the numerical algorithm to the different meshes. We increase the speed of the top wall and

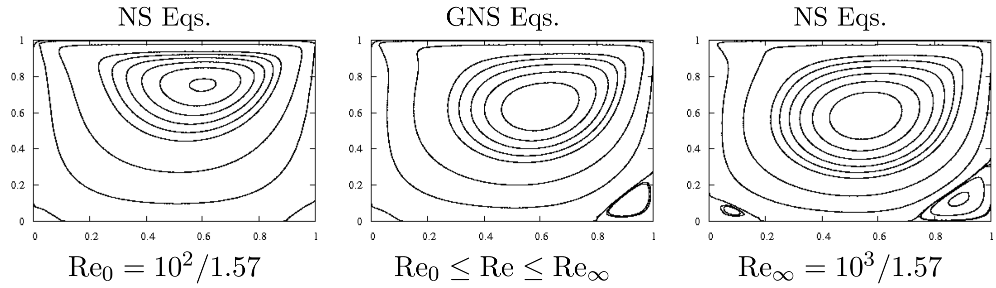


Figure 7: $U = 0.01\mathrm{ms}^{-1}$, $a = 0.144$. Contour plot of stream functions, steady solutions. Regular grid, 51×51 grid points.

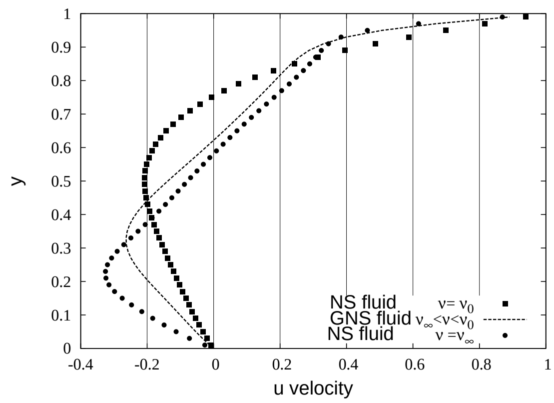
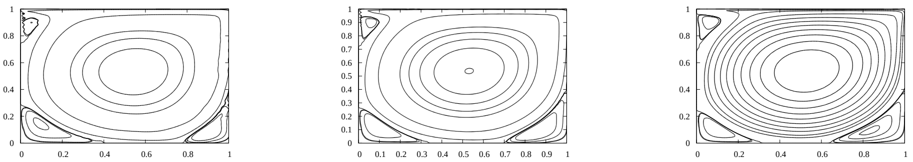


Figure 8: $U = 0.01\mathrm{ms}^{-1}$, $a = 0.144$. Distribution of u-velocity along at vertical centre line of the cavity. Regular grid, 51×51 grid points.

perform the computations using three type of meshes, two regular meshes one with 61×61 grid points and other with 81×81 grid points and the third is a non-regular grid with 51×51 grid points. In the non-regular meshes the grid point are more dense distributed near the walls. We can see that, figure 9 at high Reynolds number one needs a finner grid to capture the details of the motion. The third set of computations analyse the response of the numerical method to the variation of the parameters of the fluid. The results are shown in the figure 10.

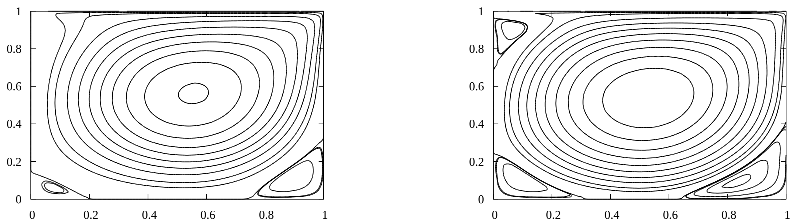
5.3 T-shape Micro-Channel

The fluid flow in microdevice are mainly charcterizate by the low Reynolds number and the large surface-to-volume ratios. As a consequence the viscous



61 × 61 grid points | 81 × 81 grid points | stretched grid

Figure 9: $U = 0.1\mathrm{ms}^{-1}$,$a = 0.144$, $\mathrm{Re}_\infty = 10^4/1.57$, $\mathrm{Re}_0 = 10^3/1.57$ GNS Eqs. Contour plot of stream function at t=200



$a = 0.644$
steady solution

$a = 0.144$
$t = 200$

Figure 10: GNS Eqs. $U = 0.1\mathrm{ms}^{-1}$ $\mathrm{Re}_\infty = 10^4/1.57$, $\mathrm{Re}_0 = 10^3/1.57$. Stretched grid, 51×51 grid points.

forces dominate fluid flow and there exists a large variation of the shear rates. As a consequence, Newtonian fluid is inadequate medel for biofluid flow throgh microdevice.

The geometry of microdevice and the flow parameter are given in the figure 11. The figure 12 give a image of the distribultion of the velocity field along the wings of device and the foot of T-shape.

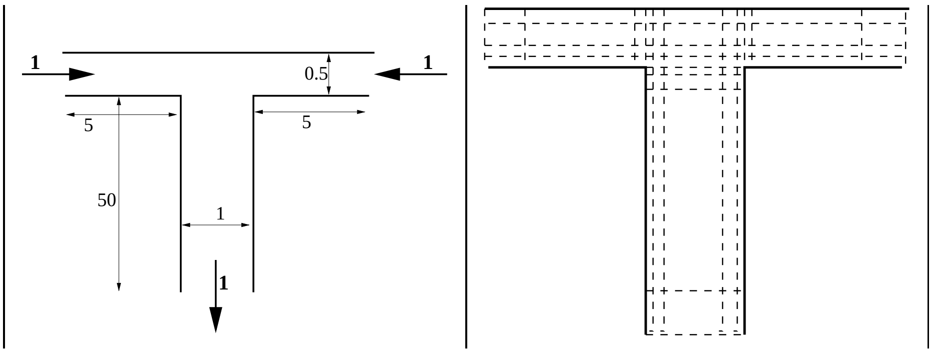


Figure 11: The geometry of T-shape microdevice, dimensionless units (left). The stretched grid with $(21, 51, 21) \times (41, 21)$ points (right). The limits of Renolds' number are $\mathrm{Re}_\infty = 200$, $\mathrm{Re}_0 = 20$

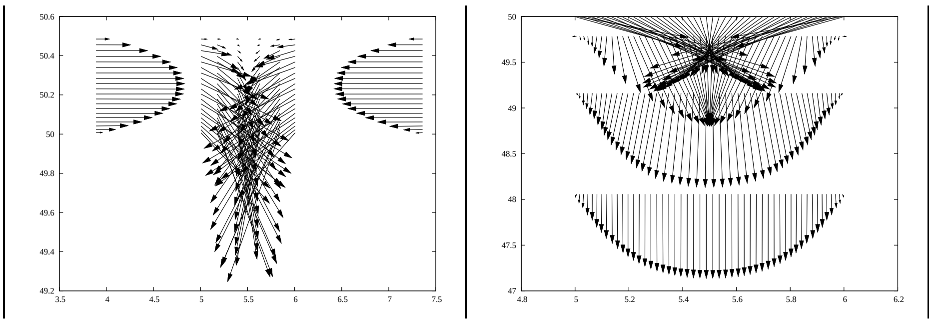


Figure 12: The distribution of the velocty field along the device.

Final Remarks

A certain advantage of our method is that there is no need to introduce artificial boundary conditions for the pressure field or supplementary boundary conditions for additional velocity field as in the projection methods or gauge methods. The preliminary numerical results prove a good agreement with the results obtained by other methods. At the present moment we do not know if it is possible to extended the method to the 3D case and this is a drawback of the method.

6 Appendix A.

We revised here some generalised Newtonian model for pseudo-plastic fluids. Such fluids are characterized by the fact that apparent viscosity is a decreasing function with respect to shear rate.

In all models μ_0 and μ_∞($\mu_0 > \mu_\infty$) are the asymptotic apparent viscosities as $\dot\gamma \to 0$ and ∞ respectively, and $\Lambda \geq 0$ is a material constant with dimension of time.

• *Powell-Eyring model*: Is an old three parameters model for the suspensions of polymer in solvents and polymer melts with low elasticity , [6], [12], [15],

$$\mu(\dot\gamma) = \mu_\infty + (\mu_0 - \mu_\infty)\frac{\sinh^{-1}\Lambda\dot\gamma}{\Lambda\dot\gamma} \tag{100}$$

• *Yeleswarapu model*: Is a model proposed in [25] for the the constitutive behavior of the blood,

$$\mu(\dot\gamma) = \mu_\infty + (\mu_0 - \mu_\infty)\frac{1 + \ln(1 + \Lambda\dot\gamma)}{1 + \Lambda\dot\gamma}. \tag{101}$$

• *Cross model*: Is a four parameters model, [13]

$$\mu(\dot\gamma) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + (\Lambda\dot\gamma)^m}. \tag{102}$$

• *Carreau-Yasuda model*: In its general form is five parameters model, [7], [24],

$$\mu(\dot\gamma) = \mu_\infty + (\mu_0 - \mu_\infty)\left(1 + (\Lambda\dot\gamma)^a\right)^{(n-1)/a}, \quad 0 < n < 1. \tag{103}$$

We note that in all previously model the constitutive function $\nu(\cdot)$ is a continuous, bounded and monotone function with respect with their argument.

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