Mathematical Modelling of Soil Erosion Process

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Technical report

1 Introduction

By erosion one means the process of detachment and transport of soil particle by different agents, especially wind and rain. Among the causes of soil erosion one can mention inappropriate agriculture practice, deforestation, overgrazing and construction activities, [5]. The studies of erosion are of interest in the scientific planning of soil and water conservation and also in the study and prevention of the surface water pollution from contaminated soil.
The mathematical modelling of the erosion processes needs physical description of the phenomenon and data measurements. The physical description provides a base for the identification of key variables that quantify the erosion process and for formulating the mathematical relations between key variables. The data measurements are necessary to evaluate the parameters in the mathematical model and to validate the model.
A survey of the scientific articles dedicated to the soil erosion indicates three main clusters of mathematical models: statistical models, partial differential equation models and cellular automata models. A model in each class is well suited to certain soil erosion type and it is for limited use. Up to now a mathematical model for general use does not exist.
In the sequel we will discuss the physical factors that affect the soil erosion, section 2, and some mathematical models widely used by practitioner, section 3. We restrict to water erosion on hillslope.

2 Soil erosion. Physical description

Among the all erosion processes we focus on water erosion on hillslope. Rainsplash and rainoff energy are the active erosive agents that produce, if ignore gully formation, two main sub-processes: interrill erosion and rill erosion.
Interrill erosion. Initially the soil particles are moved from their initial position by raindrop impact, splash erosion, and then they are transported by the overland flow. The mass of detached particle depends on the kinetic energy of the raindrops and soil texture. The runoff appears if the rainfall intensity exceeds the infiltration capacity of the soil [8], Hortonian flow. The runoff starts at the end

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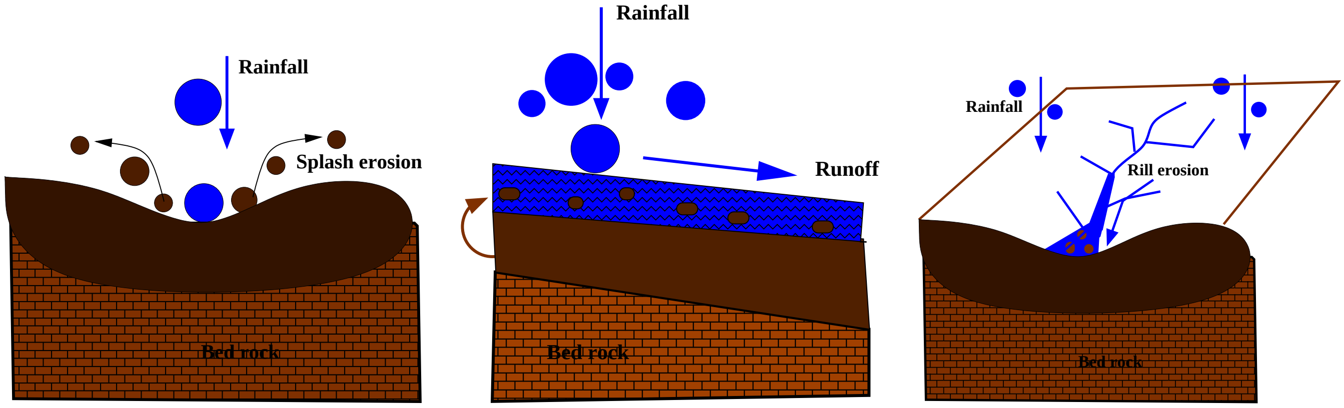

Figure 1: Three erosion sub-processes: splash erosion, runoff erosion and rill erosion.

of the process of soil saturation and after the soil depression storage spaces are filled. At the beginning it is a thin film that move slowly, it has low kinetic energy and it is incapable to detach soil particles or transport them. If the rain continues, the water depth is increasing, the water is likely to move downhill more quickly, gain kinetic energy and is able to detach soil particles and transport them. On the other hand, the accumulation of water on the soil surface protects the soil from the direct impact of raindrops, these implying a decrease of splash erosion intensity. Hence the erosion of the soil by raindrops and runoff is a complex process that needs special investigation, [13].
Rill erosion. The rill erosion involves the flow concentration in small channels, where the water depths is of the order of centimeters. The rill bed surface changes as soil is eroded, which in turn alters the hydraulics of the flow. The hydraulics is the driving force for mechanism of erosion process. Thus the process of rill erosion involves a feedback loop between flow detachment, hydraulics and sediment bed, []. The rill initialization and their distribution are strong influenced by the hydraulic condition and soil properties.
To quantify the erosion process one identifies the following key variables, processes and factors: **variables**: water depth, water velocity, kinetic energy, quantity of sediment transported; **processes**: water moving, sediment dislocation and its transport; **external media factors**: rainfall, soil.

3 A mathematical model

3.1 The universal soil loss equation

The universal soil loss equation (USLE) and its revised form (RUSLE) is a statistical erosion model designated to compute the longtime soil losses. The equation groups a series of physical and management parameters that influence the erosion rate under six factors, rainfall-rainoff erosivity factor, soil erosivity factor, slope length factor, slope steepness factor, cropping management factor and conservation practice factor, [17]. The soil loss equation is

$$\mathcal{A} = \mathcal{R} \cdot \mathcal{K} \cdot \mathcal{L} \cdot \mathcal{S} \cdot \mathcal{C} \cdot \mathcal{P}, \tag{1}$$

with the notations

$\boldsymbol{\mathcal{A}}$ - the average annual soil loss;

$\boldsymbol{\mathcal{R}}$ - the rainfall-rainoff erosivity factor. It must quantify the raindrop impact effect, runoff associated with rain and must also provide information relative to erosive force of runoff from thaw, snowmelt and irigation. It depends on the area studied and there exist maps for it for European countries (see [5]);

$\boldsymbol{\mathcal{K}}$ - the soil erosivity factor. It accounts for the influence of the properties of soil itself . Here is an empirical formula to evaluate it [5]

$$\boldsymbol{\mathcal{K}} = 0.0034 + 0.0405 \exp\left(-0.5\left(\frac{\ln D_g + 1.659}{0.7101}\right)^2\right),$$

where: D_g is the geometric mean weight diameter of the primary soil particle;

$\boldsymbol{\mathcal{L}}$ - a length factor, a recommended formula is, [5]

$$\boldsymbol{\mathcal{L}} = 1.4\left(\frac{A_s}{22.13}\right)^{0.4}$$

where A_s is specific contributing area, (conventionally taken as 50 m).

$\boldsymbol{\mathcal{S}}$ - a slope factor,

$$\boldsymbol{\mathcal{S}} = \left(\frac{\sin\beta}{0.0896}\right)1.3,$$

where β is slope angle.

$\boldsymbol{\mathcal{P}}$ - a cover and management factor. It is the ratio of soil loss from an area with specific cover and management to that from an identical area in tilled continuous fallow [7]. After the topographical factor is the second most important factor that controls the sediment loss, [5]

$$\boldsymbol{\mathcal{S}} = \exp\left(-\alpha\frac{N_{DVI}}{\xi - N_{DVI}}\right),$$

where α and β are parameters and N_{DVI} is *Normalized Difference Vegetation Index.*

$\boldsymbol{\mathcal{C}}$ The support practice factor, that takes into accont the practice to protect soil from erosion. Most ofteen is included in $\boldsymbol{\mathcal{C}}$, [5]

3.2 Physically-based erosion model

The physical-based erosion model includes a set of equations derived from balance of mass and empirical laws.
The model assumes that:
– there exists a bed of rock soil;

– the bed of rock soil is covered by a stratum of mobile soil;
– the mobile soil consists in two phases: suspended particles in water and deposited layer;
– there exists a mass exchange between suspended phase and deposited phase.
The dynamics of erosion is modelled by a differential equation that is established by considering the mass balance of water and sediment. Let ω be a representative element and let $m_s(t;\omega)$ be the mass of mobile soil in ω. The $m_s(t;\omega)$ can be considered as the sum of the mass of suspended phase, $m_s^f(t;\omega)$ and the mass of deposited phase $m_s^d(t;\omega)$

$$m_s(t;\omega) = m_s^f(t;\omega) + m_s^d(t;\omega). \tag{2}$$

Generally, the mass balance for mobile sediment can be formulated as

$$\begin{aligned} \frac{\mathrm{dm_s^f(t;\omega)}}{\mathrm{dt}} + \mathcal{J}_s(\partial\omega) &= \mathcal{M}_g^f(t;\omega) - \mathcal{M}_l^f(t;\omega) \\ \frac{\mathrm{dm_s^d(t;\omega)}}{\mathrm{dt}} &= \mathcal{M}_g^d(t;\omega) - \mathcal{M}_l^d(t;\omega) \end{aligned} \tag{3}$$

where $\mathcal{J}$ stands for the mass flux of suspended sediment through the border of ω. Since there are no other sources of mass, one has

$$\mathcal{M}_g^f(t;\omega) - \mathcal{M}_l^f(t;\omega) + \mathcal{M}_g^d(t;\omega) - \mathcal{M}_l^d(t;\omega) = 0. \tag{4}$$

The mass balance for the water flow reads as

$$\frac{\mathrm{dm_w(t;\omega)}}{\mathrm{dt}} + \mathcal{J}_w(\partial\omega) = \mathcal{P}_w(t;\omega) - \mathcal{I}_w(t;\omega) \tag{5}$$

where $\mathcal{I}_w(\partial\omega)$ and $\mathcal{P}_s(\omega)$ quantify the water infiltrated in soil and water from rain.
St. Venant equation. The water is moving on the surface of the hillslope as a layer of depth h with velocity $\boldsymbol{v}$. The mass balance equation is

$$\frac{\partial h}{\partial t} + \nabla \boldsymbol{q} = f, \tag{6}$$

where f is net water gain intensity and the density of the mass flux is given by

$$\boldsymbol{q} = h\boldsymbol{v}.$$

Sediment transport. The mass balance for sediment is similar to mass balance of water

$$\frac{\partial hc_s}{\partial t} + \nabla \boldsymbol{q c_s} = e/\rho_s \tag{7}$$

where c_s is volumetric sediment concentration, e erosion rate, and ρ_s is specific mass of sediment.
As an illustrative example of the previously general formulation we present a model for hillslope erosion, [1].
Erosion on planar hillslope without rill and in absence of infiltration, 1D flow. The velocity is empirically related to water depth by

$$v = \alpha h^m. \tag{8}$$

α is empirically related to topographic slope, S, by $\alpha = cS^{0.5}$ (Chezy formula) or $\alpha = S^{0.5}/n$ (Maning formula).
Erosion is related to rainfall erosion, e_r, and overland erosion, e_f as

$$e = e_r + e_f$$

The rainfall erosion depends on rain intensity, r and is given by

$$e_r = \chi r^n. \tag{9}$$

The overland erosion

$$e_f = \sigma(T_c - q_s), \tag{10}$$

where σ is transfer rate coefficient, T_c is the transport capacity of the flow and q_s is sediment mass flux density

$$q_s = \rho_s c_s q$$

The transport capacity is related to shear stress by

$$T_c = \eta(\tau - \tau_c)^l, \tag{11}$$

where η is a dimensional coefficient, τ and τ_c are the shear stress and its critically values, respectively. The shear stress exerted by the flow is given by:

$$\tau = \gamma h S, \tag{12}$$

where γ is the specific weight of the water.

4 Cellular automata models of erosion soil

Cellular automata (CA) provide an attractive alternative to physical-based models for simulating the dynamics of soil erosion processes through simple local interaction rules. Some CA models have been proposed to simulate water runoff, [4], sediment erosion and transport, [3], [10], rill formation and rill erosion, [9], see also [6] and [15].
Briefly, a cellular automaton is a kind of dynamical system which is discrete in time and space. Its basic definition includes a cell and its neighbors (the neighborhood of a cell may be defined in several ways). Each cell is characterized by a number of features that, together, define the state of the cell, and by certain parameters that remain unchanged in the course of the evolution of the process (e.g., for CA that model erosion, the altitude of the cell, or the roughness of the soil in the cell). Each iteration defines a new state of the cell by considering the states of the neighboring cells and according to some specific rules. Hence the basic elements of a CA are: the neighborhood of a cell, the state and the global parameters, and the rule that gives the change of the state at every time step.
We present two different CA models for erosion, found in the literature.
In [10] a model for the erosion process, named SCAVATU, is constructed. The main erosion factor is the water coming from rain. The neighborhoods used are Von Neumann neighborhoods. The model is improved in [11], where hexagonal cells are considered. We present here the main ideas of [11].

The cellular automaton SCAVATU is given by

$$SCAVATU = \langle R^2,\, X,\, S,\, P, \sigma,\, \gamma \rangle$$

where

- $$R^2 = \{(x, y) | x, y \in \mathbf{N}, 0 \le x < l_x,\, 0 \le y < l_y\}$$
  that is the set of points with coordinates given by entire numbers in the region where the process evolves;
- X is the set of the coordinates of the centers of the cells from the neighborhood of cell 0 (hexagonal cells);
- S is the set of substates of the finite automaton represented by the cell 0 and
  $$S = S_a \times S_{wd} \times S_T \times S_E \times S^w_{out} \times S^T_{out}$$
  or
  $$S = S_a \times S_{wd} \times S_T \times S_E \times S^{(w)}_{in} \times S^T_{in},$$
  where S_a is the altitude of the cell, S_{wd} - the water depth, S_T - the sediment transport, S_E - water energy, S^w_{out} - water outflow, S^w_{in} - water inflow, S^T_{out} - sediment transport outflow, S^T_{in} - sediment transport inflow;
- P is the set of global parameters of the CA, like: P_a - the apothem of the hexagon, P_{ts} - the time step, P_f - a friction coefficient;
- $\sigma : S^7 \mapsto S$ is the transition function. The hydrodynamic model is determined by I_1, $I_2\, I_3$, where I_1 describes the flux of water and the transport of soil from the central cell to the others, I_2 gives the water depth of the cell, and the height of the deposited solid material, while I_3 describes the changes in the total energy of the cell.
- $\gamma : \mathbb{N} \mapsto S_{wd} \times S_E$ specifies the variation of the water height (rain water) and of its energy, at any time step of the CA (the evolution in time of the rain water input).

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The changes of the substates are defined as follows.
For I_1,

$$I_1 : S_a^7 \times S_{wd}^7 \times S_T^7 \times S_E \mapsto \left[S^{(w)}_{out}\right]^6 \times \left[S^{(w)}_I\right]^6.$$

The hydrodynamic flux is determined from energetic considerations. That is the total (energetic) charge of a cell is defined as the sum of the altitude, S_a, the water depth, S_{wd}, and the "kinetic height" $v^2/2g$ (where v is the velocity, and g is the gravitational acceleration).
The idea is that the state of a neighborhood evolves towards an equilibrium situation - that is the difference between the total charge of the central cell and the partial charge of the neighbor cells tends to be minimized. An algorithm proposed by Gregorio and Serra in [12] is used.

The local interaction I_2,

$$I_2 : S_{wd} \times \left[S_{in}^{(w)}\right]^6 \times \left[S_{out}^{(w)}\right]^6 \times \left[S_{in}^{(T)}\right]^6 \times \left[S_{out}^{(T)}\right]^6 \mapsto S_{wd} \times S_T$$

determines the values of the sub-states water depth S_{wd}^{new}, and solid material height, S_T^{new}. For any cell, these values are equal to the algebraic sum of the values at the preceding step and the flux entering from the other cells, minus the flux that gets out from the cell:

$$S_{wd}^{new} = S_{wd} + \sum_{i=1}^{6} S_{in,i}^{(w)} - \sum_{i=1}^{6} S_{out,i}^{(w)}$$

$$S_T^{new} = S_T + \sum_{i=1}^{6} S_{in,i}^{(T)} - \sum_{i=1}^{6} S_{out,i}^{(T)}.$$

The interaction I_3:

$$I_3 : S_a^7 \times S_{wd} \times \left[S_{in}^{(w)}\right]^6 \times \left[S_{out}^{(w)}\right]^6 \times S_E^7 \mapsto S_E,$$

determines the value that the energy takes at a time moment.
The new value of the substate "hydrodynamic energy" is given by

$$S_{E,0}^{new} =$$

$$= \max \left\{ \left(S_{wd,0}^{new}\right)^2,\ S_{E,0} + \sum_{i=1}^{6} \left[S_{in,i}^{(w)} \cdot (H_i - S_{a,0} - S_{wd,0})\right] - \right.$$

$$\left. - \sum_{i=1}^{6} \left[S_{in,i}^{(w)} \cdot r_0\right] - P_f + h^2 \right\}$$

where the energetic "slope" from the central cell is given by $r_0 = S_{E,0}/S_{wd,0}$, and h is the rainfall height.
We remark that the previous form of SCAVATU model [10] takes into account also the vegetation density.
In the CA model proposed in [14] the main sub-states of the cell are the water content and the sediment deposed in the cell. The mass balance for water for a given cell is given by:

$$R + \triangle Q - I + D = 0$$

where R is the received rainfall amount of water, $\triangle Q$ is the net discharge flux, I -the infiltration loss, and D - the interception and depression loss.
The flow rate V between two adjacent cells is given by

$$V = h^{2/3} J^{1/2} / n$$

$(m \cdot s^{-1})$ where h is the water depth, J is the water surface slope and n is Manning's roughness coefficient. The interchanged water amount between two adjacent spatial cells $Q\,(m^3)$, in the interval Δt, is given by

$$Q = d \times V \times h \times \Delta t.$$

The flow direction is such that the water is always transported from a spatial cell with higher potential head to another neighbor cell with low potential head.
The sediment S (kg) transport is described by the law $S = kQV^q$.
Such models of erosion may prove very useful in the study of dispersion of some pollutants contained in restrained geographical areas (like residual substances from some mining plant), by the erosion and transport of the soil, due to rainfall. However, an application of these models to concrete problems would require the good knowledge of the geography of the zone (GIS must be used), knowledge of the properties of the soil, of the vegetation density, of the concentrations of the pollutants in the contaminated zone. Without these elements, any models and any computer program will give not practical but only theoretical results.

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