1

Analitical Solutions for Reentry of the Spatial Vehicles in Atmosphere

MIHAI POPESCU

Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, P.O. Box 1-24, RO-010145, Bucharest, Romania.

Summary. In the dynamics of spatial vehicles reentry into the earth atmosphere, an important factor is the existence of aerodynamic forces. The mathematical model adopted regarding optimal atmosphere re-entry considers the lift - resistance ratio as being constant. Actually, the aerodynamic coefficients vary with the flight velocity and altitude, a hypothesis used in this study. Thus, the system that represents the motion equations of atmosphere re-entry changes into an equivalent system with small parameters. The analytical solution of this system is expressed through the variable defined by atmospheric density.

Keywords: Motion equations, aerodynamic coefficients, atmospheric density, small parameters.

1 Introduction

In the problem of optimum atmosphere re-entry one considers that the ratio of the lift forces and resistance are constant. That is only a unanimously recognized hypothesis that lessens the difficulty of solving optimization problems. Mention should be made that one can find of particular interest the case when - closer to the real phenomenon - the coefficients of the lift and resistance forces are functions of the flight velocity and altitude.

The equations governing the vehicle entry taken as a mass point in the rotational atmosphere, whose lift and resistance coefficients arbitrarily given as velocity and altitude functions are solved by using series.

The only assumption made in this respect is that the altitude variation compared to the earth radius is small.

Out of the studies carried out, one has found that for constant motion lift and resistance, the solution does not exactly verify the numerical results. The literature provides numerous publications regarding the approximation of the solutions with those corresponding to some equations governing the vehicle entry with constant aerodynamic coefficients.

Wand and Chu have studied the case of variable lift where the trajectory angle is smaller and the lift coefficient is represented by $\mathbf{C}_P = \mathbf{C}_{P_E} - \xi\,\mathbf{X}^n$ where $\mathbf{C}_{P_E}$ is the lift coefficient at entry, ξ and n are the two lift parameters and $\mathbf{X}$ is a proportional dimension with atmospheric density. The problem of integration has been approached by assuming that the solution for the flight trajectory given by the angle θ can be developed in series of the form (1)

$$\theta = \sum a_{i\,j}\, x^i\ (\log\, \mathbf{X}/\mathbf{X}_E)^i \tag{1}$$

These solutions are obtained by using the following hypotheses:

- The trajectory angle is small,
- The lift coefficient assumed for a special form varies with altitude,
- One neglects the lift parameter n that might complicate the problem a lot.

2 Dynamical reentry

Further down we mean to integrate the vehicle motion equations representing entry in the three-dimensional case to be followed by particularization likely to yield the solutions for the bidimensional case as well. It will be possible to obtain the analytical solutions by using series development with respect to a small parameter allowing us to neglect higher order terms.

Thus, the aerodynamic forces coefficients will be functions of variables $\mathbf{v}$, $\mathbf{r}$, velocity and vehicle altitude.

$$\begin{aligned} \mathbf{C}_R &= \mathbf{C}_R(\bar{\mathbf{v}},\, \bar{\mathbf{r}}) \\ \mathbf{C}_P &= \mathbf{C}_P(\bar{\mathbf{v}},\, \bar{\mathbf{r}}) \end{aligned} \tag{2}$$

We shall write $\bar{t}$, $\bar{\mathbf{v}}$, $\bar{\mathbf{r}}$ for time, velocity and flight altitude in order to discriminate them from the adimensional variables t, $\mathbf{v}$, $\mathbf{r}$ underlying the system of motion equations.

We resume the motion equations system:

$$\begin{cases} m\, \dfrac{d\,\bar{\mathbf{v}}}{d\,\bar{t}} = -\dfrac{\bar{\rho}}{2}\, S\, \bar{\mathbf{v}}^2\, \mathbf{C}_R + m\, \mathbf{g}\, \sin\,\theta \\ -m\, \bar{\mathbf{v}}\, \dfrac{d\,\theta}{d\,\bar{t}} = \dfrac{\bar{\rho}}{2}\, S\, \bar{\mathbf{v}}^2\, \mathbf{C}_P\, \cos\,\gamma - m\, \left(\mathbf{g} - \dfrac{\bar{\mathbf{v}}^2}{\bar{\mathbf{R}} + \bar{\mathbf{r}}}\right) \cos\,\theta \\ \cos\,\theta\, \dfrac{d\,n}{d\,\bar{t}} = \dfrac{\bar{\rho}}{2}\, S\, \bar{\mathbf{v}}\, \mathbf{C}_P\, \sin\,\gamma - \dfrac{\bar{\mathbf{v}}}{\bar{\mathbf{R}} + \bar{\mathbf{r}}}\, \tan\,\varphi\, \cos^2\,\theta\, \cos\,\eta \\ \dfrac{d\,\varphi}{d\,\bar{t}} = \dfrac{\bar{\mathbf{v}}}{\bar{\mathbf{R}} + \bar{\mathbf{r}}}\, \cos\,\theta\, \sin\,\eta \\ \dfrac{d\,\bar{\mathbf{r}}}{d\,\bar{t}} = -\bar{\mathbf{v}}\, \sin\,\theta \\ \dfrac{d\,\lambda}{d\,\bar{t}} = \dfrac{\bar{\mathbf{v}}}{\bar{\mathbf{R}} + \bar{\mathbf{r}}}\, \cos\,\theta\, \dfrac{\cos\,n}{\cos\,\varphi} \end{cases} \tag{3}$$

If we work out the adimensionalisation

$$\mathbf{r} = \frac{\bar{\mathbf{r}}}{\bar{\mathbf{R}}}\,, \quad t = \frac{\bar{t}}{\bar{T}}\,, \quad \mathbf{v} = \frac{\bar{T}\, \bar{\mathbf{v}}}{\bar{\mathbf{R}}}\,, \quad \rho = \frac{S\, \bar{\mathbf{R}}\, \bar{\rho}}{m}\,, \quad \bar{T} = \left(\mathbf{R}^3/f\, M\right)^{\frac{1}{3}} \tag{4}$$

and by taking into account that

$$\mathbf{g} = f\, \frac{M}{\bar{\mathbf{R}} + \bar{\mathbf{r}}} \tag{5}$$

where f is the universal gravity constant, M the Earth mass, system (3) is written adimensionally as

$$\begin{cases} \dfrac{d\,\mathbf{v}}{d\,t} = -\dfrac{1}{2}\,\mathbf{C}_R(\mathbf{v},\,\mathbf{r})\,\rho\,\mathbf{v}^2 + \dfrac{\sin\,\theta}{(1+\mathbf{r})^2} \\ -\mathbf{v}\,\dfrac{d\,\theta}{d\,t} = \dfrac{1}{2}\,\mathbf{C}_P(\mathbf{v},\,\mathbf{r})\,\rho\,\mathbf{v}^2\,\cos\,\gamma - \dfrac{\cos\,\theta}{(1+\mathbf{r})^2} - \dfrac{\mathbf{v}^2\,\cos\,\theta}{1+\mathbf{r}} \\ \dfrac{d\,n}{d\,t} = \dfrac{1}{2}\,\rho\,\mathbf{v}\,\mathbf{C}_P(\mathbf{v},\,\mathbf{r})\,\dfrac{\sin\,\gamma}{\cos\,\theta} - \mathbf{v}\,\tan\,\varphi\,\cos^2\,\theta\,\cos\,\eta \\ \dfrac{d\,\varphi}{d\,t} = \mathbf{v}\,\cos\,\theta\,\sin\,\eta \\ \dfrac{d\,\lambda}{d\,t} = \mathbf{v}\,\cos\,\theta\,\dfrac{\cos\,n}{\cos\,\varphi} \\ \dfrac{d\,\mathbf{r}}{d\,t} = -\mathbf{v}\,\sin\,\theta \end{cases} \tag{6}$$

or by passing to the variable $\mathbf{r}$

$$\frac{d\,\mathbf{v}}{d\,t} = \frac{d\,\mathbf{v}}{d\,\mathbf{r}}\,\frac{d\,\mathbf{r}}{d\,t} = -\mathbf{v}\,\sin\,\theta\,\frac{d\,\mathbf{v}}{d\,\mathbf{r}} = \frac{d}{d\,\mathbf{r}}\left(\frac{\mathbf{v}^2}{2}\right)\sin\,\theta = -\frac{1}{2}\,\frac{d\,\mathbf{v}^2}{d\,\mathbf{r}}\,\sin\,\theta$$

By taking into consideration the approximation $1+\mathbf{r} \approx 1$, system (6) becomes

$$\begin{cases} \dfrac{d\,\mathbf{v}^2}{d\,\mathbf{r}} - \dfrac{\mathbf{C}_R(\mathbf{v},\,\mathbf{r})}{\sin\,\theta}\,\rho\,\mathbf{v}^2 + 2 = 0 \\ \dfrac{d\,(\cos\,\theta)}{d\,\mathbf{r}} + \left(1 - \dfrac{1}{\mathbf{v}^2}\right)\cos\,\theta + \dfrac{1}{2}\,\mathbf{C}_P(\mathbf{v},\,\mathbf{r})\,\rho\,\cos\,\gamma = 0 \\ \dfrac{d\,n}{d\,\mathbf{r}} = -\dfrac{1}{2}\,\rho\,\mathbf{C}_P(\mathbf{v},\,\mathbf{r})\,\dfrac{\sin\,\gamma}{\cos\,\theta\,\sin\,\theta} + \tan\,\varphi\,\dfrac{\cos\,\theta}{\sin\,\theta}\,\cos\,\eta \\ \dfrac{d\,\varphi}{d\,\mathbf{r}} = -\dfrac{\cos\,\theta}{\sin\,\theta}\,\sin\,\eta \\ \dfrac{d\,\lambda}{d\,\mathbf{r}} = -\dfrac{\cos\,\theta}{\sin\,\theta}\,\dfrac{\cos\,\eta}{\cos\,\varphi} \end{cases} \tag{7}$$

From the variation law adopted for atmospheric density $\rho = \rho_0\,e^{-\beta\,\mathbf{r}}$ it results

$$\log\,\frac{\rho}{\rho_0} = -(\beta\,\mathbf{R})\,\mathbf{r}$$

3 Integration of the system with small parameters

By writing $\varepsilon = \dfrac{1}{\beta\,\mathbf{r}}$, which owing to its very low value cannot be taken as a small parameter, one obtains

$$\begin{cases} \dfrac{d\,\mathbf{v}^2}{d\left(\log\,\dfrac{\rho}{\rho_0}\right)} - \varepsilon\left[2 - \dfrac{\mathbf{C}_R(\mathbf{v},\,\rho)\,\rho\,\mathbf{v}^2}{\sin\,\theta}\right] = 0 \\ \dfrac{d\,(\cos\,\theta)}{d\left(\log\,\dfrac{\rho}{\rho_0}\right)} - \varepsilon\left[\left(1 - \dfrac{1}{\mathbf{v}^2}\right)\cos\,\theta + \mathbf{C}_P(\mathbf{v},\,\rho)\,\cos\,\gamma\,\dfrac{\rho}{2}\right] = 0 \end{cases} \tag{8}$$

$$\begin{cases} \dfrac{d\,n}{d\left(\log\, \dfrac{\rho}{\rho_0}\right)} - \varepsilon\, \left(\dfrac{\rho}{2}\, \dfrac{\mathbf{C}_P(\mathbf{v},\, \rho)\, \sin\, \gamma}{\cos\, \theta\, \sin\, \theta} - \tan\, \varphi\, \dfrac{\cos\, \theta}{\sin\, \theta}\, \cos\, \eta \right) = 0 \\ \dfrac{d\,\varphi}{d\left(\log\, \dfrac{\rho}{\rho_0}\right)} - \varepsilon\, \dfrac{\cos\, \theta}{\sin\, \theta}\, \sin\, \eta = 0 \\ \dfrac{d\,\lambda}{d\left(\log\, \dfrac{\rho}{\rho_0}\right)} - \varepsilon\, \dfrac{\cos\, \theta}{\sin\, \theta}\, \dfrac{\cos\, \eta}{\cos\, \varphi} = 0 \end{cases}$$

As $d\left(\log\, \dfrac{\rho}{\rho_0}\right) = d\,(\log\, \rho)$ the system (8) can be written as

$$\begin{cases} \dfrac{d\,\mathbf{v}^2}{d\,(\log\, \rho)} - \varepsilon\, \left[2 - \dfrac{\mathbf{C}_R(\mathbf{v},\, \rho)\, \rho\, \mathbf{v}^2}{\sin\, \theta} \right] = 0 \\ \dfrac{d\,(\cos\, \theta)}{d\,(\log\, \rho)} - \varepsilon\, \left[\left(1 - \dfrac{1}{\mathbf{v}^2}\right)\, \cos\, \theta + \mathbf{C}_P(\mathbf{v},\, \rho)\, \dfrac{\rho}{2}\, \cos\, \gamma \right] = 0 \\ \dfrac{d\left(\dfrac{1}{\cos\, \theta}\right)}{d\,(\log\, \rho)} - \varepsilon\, \sqrt{\left(\dfrac{1}{\cos\, \theta}\right)^2 - 1}\, \left(\dfrac{\rho}{2}\, \dfrac{\mathbf{C}_P(\mathbf{v},\, \rho)\, \sin\, \gamma}{\cos\, \theta\, \sin\, \theta} \left(\dfrac{1}{\cos\, \theta}\right) - \right. \\ \qquad\qquad \left. - \tan\, \varphi\, \dfrac{\cos\, \theta}{\sin\, \theta}\, \cos\, \eta \right) = 0 \\ \dfrac{d\,\,(\tan\, \varphi)}{d\,(\log\, \rho)} - \varepsilon\, \dfrac{\cos\, \theta}{\sin\, \theta}\, \sin\, \eta\, \left(1 + \tan^2 \varphi\right) = 0 \\ \dfrac{d\,\lambda}{d\,(\log\, \rho)} - \varepsilon\, \dfrac{\cos\, \theta}{\sin\, \theta}\, \dfrac{\cos\, \eta}{\cos\, \varphi} = 0 \end{cases} \tag{9}$$

We consider the first two equations of the system that will be written as

$$\left(\frac{d\,\mathbf{v}^2}{d\,(\log\, \rho)} - 2\, \varepsilon \right)\, \sin\, \theta = -\varepsilon\, \mathbf{C}_R(\mathbf{v},\, \rho)\, \rho\, \mathbf{v}^2$$

or by squaring

$$\left(\frac{d\,\mathbf{v}^2}{d\,(\log\, \rho)} - 2\, \varepsilon \right)\, (1 - \cos^2 \theta) = \varepsilon^2\, \left[\, \mathbf{C}_R(\mathbf{v},\, \rho)\, \rho\, \mathbf{v}^2\, \right] \tag{10}$$

The second equations is written under a convenient form as

$$\mathbf{v}^2\, \left[\frac{d\,(\cos\, \theta)}{d\,(\log\, \rho)} - \varepsilon\, \cos\, \theta - \varepsilon\, \mathbf{C}_P(\mathbf{v},\, \rho)\, \frac{\rho}{2}\, \cos\, \gamma \right] = -\varepsilon\, \cos\, \theta \tag{11}$$

As ε is a small parameter, we shall use the series development solutions under the form

$$\begin{aligned} \mathbf{v} &= f_0 + \varepsilon\, f_1 + \varepsilon^2\, f_2 + \varepsilon^3\, f_3 + \,\dots \\ \cos \theta &= g_0 + \varepsilon\, g_1 + \varepsilon^2\, g_2 + \varepsilon^3\, g_3 + \,\dots \end{aligned} \tag{12}$$

that shall satisfy the initial conditions

$$\begin{aligned} \mathbf{v}(\rho_i) &= v_i \\ \cos(\varphi_i) &= \cos\, \theta_i \end{aligned} \qquad\qquad i = 1,\, 2,\, \dots \tag{13}$$

Hence, then initial conditions become

$$
\begin{aligned}
f_0(\rho_i) &= v_i & f_n(\rho_i) &= 0 \; n \neq 0 \\
g_0(\rho_i) &= \cos\,\theta_i & g_n(\rho_i) &= 0 \; n \neq 0
\end{aligned} \tag{14}
$$

so that the aerodynamic coefficients become

$$
\begin{aligned}
\mathbf{C}_R(\mathbf{v},\,\rho) &= \mathbf{C}_R\,\left[f_0 + \varepsilon\, f_1 + \varepsilon^2\, f_2 + \varepsilon^3\, f_3 + \,\ldots,\,\rho\right] \\
\mathbf{C}_P(\mathbf{v},\,\rho) &= \mathbf{C}_P\,\left[g_0 + \varepsilon\, g_1 + \varepsilon^2\, g_2 + \varepsilon^3\, g_3 + \,\ldots,\,\rho\right]
\end{aligned} \tag{15}
$$

or by developing a Taylor series, we have

$$
\begin{aligned}
\mathbf{C}_R(\mathbf{v},\,\rho) = \mathbf{C}_R(f_0,\,\rho) &+ \varepsilon\,\frac{\partial\,\mathbf{C}_R}{\partial\,\mathbf{v}}\,(f_0,\,\rho)\,f_1 + \\
&\varepsilon^2\,\left[\frac{\partial\,\mathbf{C}_R}{\partial\,\mathbf{v}}\,(f_0,\,\rho)\,f_2 + \frac{\partial^2\,\mathbf{C}_R}{\partial\,\mathbf{v}^2}\,(f_0,\,\rho)\,f_1^2\right] + \ldots \\
\mathbf{C}_P(\mathbf{v},\,\rho) = \mathbf{C}_P(g_0,\,\rho) &+ \varepsilon\,\frac{\partial\,\mathbf{C}_P}{\partial\,\mathbf{v}}\,(g_0,\,\rho)\,g_1 + \\
&\varepsilon^2\,\left[\frac{\partial\,\mathbf{C}_P}{\partial\,\mathbf{v}}\,(g_0,\,\rho)\,g_2 + \frac{\partial^2\,\mathbf{C}_P}{\partial\,\mathbf{v}^2}\,(g_0,\,\rho)\,g_1^2\right] + \ldots
\end{aligned} \tag{16}
$$

By using equations (11) out of the identification it follows that the terms that do not contain ε are null, which means that there are satisfied the conditions

$$
\begin{cases}
\left(1 - g_0^2\right)\,\left(\dfrac{d\,f_0^2}{d\,\log\,\rho}\right) = 0 \\
f_0\,\dfrac{d\,g_0}{d\,\log\,\rho} = 0
\end{cases} \tag{17}
$$

From the first equation of (9) one deduces that

$$
-2\,\sin\,\theta\,\left[\mathbf{v}\,\frac{d\,f_1}{d\,\log\,\rho} - 1\right] = \mathbf{C}_R(\mathbf{v},\,\rho)\,\rho\,\mathbf{v}^2 \tag{18}
$$

But

$$
\begin{aligned}
\mathbf{v} &= v_i + O(\varepsilon^n) \\
\cos\,\theta &= \cos\,\theta_i + O(\varepsilon^n)
\end{aligned} \qquad n \geq 1 \tag{19}
$$

The equation (18) becomes

$$
-2\,\sin\,\theta_i,\left[v_i\,\frac{d\,f_1}{d\,\log\,\rho} - 1\right] = \mathbf{C}_R(v_i,\,\rho)\,\rho\,v_i^2 \tag{20}
$$

By developing the initial calculations one obtains

$$
v_i^2\,\left[\frac{d\,g_1}{d\,\log\,\rho} - \cos\,\theta_i - \mathbf{C}_P(v_i,\,\rho)\,\frac{\rho}{2}\,\cos\,\gamma\right] = -\cos\,\theta_i \tag{21}
$$

By working-out the calculations we find

$$
\begin{aligned}
f_1 &= \frac{1}{v_i}\,\log\,\frac{\rho}{\rho_i} - \frac{v_i}{2\,\sin\,\theta_i}\int\limits_{\rho_i}^{\rho}\rho\,\mathbf{C}_R(v_i,\,\rho)\,d\,\rho \\
g_1 &= \left(1 - \frac{1}{v_i^2}\right)\,\cos\,\theta_i\,\log\,\frac{\rho}{\rho_i} + \frac{1}{2}\int\limits_{\rho_i}^{\rho}\mathbf{C}_P(v_i,\,\rho)\,\cos\,\gamma\,d\,\rho
\end{aligned} \tag{22}
$$

Out of relations

$$\begin{aligned}
\frac{d\,f_2}{d\,\log\,\rho} &= -\frac{1}{2\,\sin\,\theta_i}\left[2\,\sin\,\theta_i + \frac{\cos\,\theta_i}{\sin^2\,\theta_i}\,\mathbf{C}_R(v_i,\,\rho)\,\rho\,v_i^3 + \right.\\
&\qquad\left. + \left(\mathbf{C}_R(v_i,\,\rho)\,\rho\,v_i^2 + \frac{\partial\,\mathbf{C}_R(v_i,\,\rho)}{\partial\,\mathbf{v}}\,\rho\,v_i^3\right)\,f_1\right]\\
\frac{d\,g_2}{d\,\log\,\rho} &= g_1 + \frac{\partial\,\mathbf{C}_P(v_i,\,\rho)}{\partial\,\mathbf{v}}\,\frac{\rho}{2}\,f_1\,\cos\,\gamma -\\
&\qquad -\frac{2}{v_i}\,f_1\,\left[\frac{d\,g_1}{d\,\log\,\rho} - \cos\,\theta_i - \frac{\rho}{2}\,\mathbf{C}_P(v_i,\,\rho)\,\cos\,\gamma\right] - \frac{g_1}{v_i^2}
\end{aligned}\tag{23}$$

one obtains the following expressions

$$\begin{aligned}
f_2 &= -\log\,\frac{\rho}{\rho_i} - \frac{\cos\,\theta_i}{\sin^3\,\theta_i}\,v_i^3\int_{\rho_i}^{\rho}\mathbf{C}_R(v_i,\,\rho)\,d\,\rho + v_i\int_{\rho_i}^{\rho}\mathbf{C}_R(v_i,\,\rho)\,\log\,\frac{\rho}{\rho_i}\,d\,\rho -\\
&\frac{v_i^3}{\sin\,\theta_i}\int_{\rho_i}^{\rho}\mathbf{C}_R(v_i,\,\rho)\,d\,\rho\int_{\rho_i}^{\rho}\rho\,\mathbf{C}_R(v_i,\,\rho)\,d\,\rho - v_i^2\int_{\rho_i}^{\rho}\frac{\partial\,\mathbf{C}_R(v_i,\,\rho)}{\partial\,\mathbf{v}}\,\log\,\frac{\rho}{\rho_i}\,d\,\rho -\\
&\frac{v_i^4}{2\,\sin\,\theta_i}\int_{\rho_i}^{\rho}\frac{\partial\,\mathbf{C}_R(v_i,\,\rho)}{\partial\,\mathbf{v}}\,d\,\rho\int_{\rho_i}^{\rho}\rho\,\mathbf{C}_R(v_i,\,\rho)\,d\,\rho\\
g_2 &= \left[\left(1 - \frac{1}{v_i^2}\right)\,\cos\,\theta_i + \frac{2}{v_i^4}\,\cos\,\theta_i\right]\int_{\rho_i}^{\rho}\log\,\frac{\rho}{\rho_i}\,d\,\rho +\\
&\frac{\cos\,\gamma}{2\,v_i}\int_{\rho_i}^{\rho}\frac{\partial\,\mathbf{C}_P(v_i,\,\rho)}{\partial\,\mathbf{v}}\,d\,\rho + \frac{\cos\,\gamma}{2\,v_i^2}\int_{\rho_i}^{\rho}\mathbf{C}_P(v_i,\,\rho)\,\log\,\frac{\rho}{\rho_i}\,d\,\rho +\\
&\frac{\cos\,\gamma}{2}\int_{\rho_i}^{\rho}d\,\rho\int_{\rho_i}^{\rho}\rho\,\mathbf{C}_P(v_i,\,\rho)\,\cos\,\gamma\,d\,\rho - \frac{v_i\,\cos\,\gamma}{2\,\sin\,\theta_i}\int_{\rho_i}^{\rho}\frac{\partial\,\mathbf{C}_P}{\partial\,\mathbf{v}}\,d\,\rho\int_{\rho_i}^{\rho}\rho\,\mathbf{C}_P(v_i,\,\rho)\,d\,\rho\\
&-\frac{1}{\sin\,\theta_i}\,\left(\frac{\cos\,\theta_i}{v_i^2} - \cos\,\gamma\right)\int_{\rho_i}^{\rho}\rho\,\mathbf{C}_P(v_i,\,\rho)\,d\,\rho
\end{aligned}\tag{24}$$

By analyzing the system (9) one finds that for small values of φ and high values close to $\dfrac{\eta}{2}$ of η the term $\tan\,\varphi\,\cos\,\eta$ may be neglected in comparison to $\dfrac{\rho}{2}\,\mathbf{C}_P(\mathbf{v},\,\rho)\,\sin\,\gamma\,\left(\dfrac{1}{\cos\,\eta}\right)$ so that by squaring both system members one obtains the following system which will also be solved through series developments according to the small parameter ε.

$$\begin{cases}
4\,\sin^2\theta\,\cos^2\theta\,\left[\dfrac{d\left(\dfrac{1}{\cos\,\eta}\right)}{d\,\log\,\rho}\right]^2 = \\
\qquad\qquad \varepsilon\,\left[\left(\dfrac{1}{\cos\,\eta}\right)^2 - 1\right]\,\rho^2\,\mathbf{C}_P(\mathbf{v},\,\rho)\,\left(\dfrac{1}{\cos\,\eta}\right)^2\,\sin^2\gamma\\
\left(\dfrac{1}{\cos\,\eta}\right)^2\,\sin^2\theta\,\left[\dfrac{d\,(\tan\,\varphi)}{d\,\log\,\rho}\right] = \varepsilon^2\cos^2\theta\,\left[\left(\dfrac{1}{\cos\,\eta}\right)^2 - 1\right]\,(1+\tan^2\varphi)^2\\
\sin^2\theta\,\left(\dfrac{1}{\cos\,\eta}\right)\,\left[\dfrac{d\,\lambda}{d\,\log\,\rho}\right]^2 = \varepsilon^2\cos^2\theta\,(1+\tan^2\varphi)
\end{cases}\tag{25}$$

Let the unknown functions given by expression

$$\begin{cases} \dfrac{1}{\cos\,\eta} = h_0 + \varepsilon\, h_1 + \varepsilon^2\, h_2 + \dots \\ \tan^2\varphi = \rho_0 + \varepsilon\, \rho_1 + \varepsilon^2\, \rho_2 + \dots \\ \lambda = q_0 + \varepsilon\, q_1 + \varepsilon^2\, q_2 + \dots \end{cases} \tag{26}$$

By imposing the initial conditions

$$\begin{cases} \left(\dfrac{1}{\cos\,\eta}\right)(\rho_i) = \dfrac{1}{\cos\,\eta_i} \\ (\tan\,\varphi)\,(\rho_i) = \tan\,\varphi_i \\ \lambda\,(\rho_i) = \lambda_i \end{cases} \tag{27}$$

it follows that by working out the calculations

$$\begin{cases} h_0 = \dfrac{1}{\cos\,\eta_i} \\ p_0 = \tan\,\varphi_i \\ q = \lambda_i \end{cases} \tag{28}$$

or by writing

$$4\,\sin^2\theta\,\cos^2\theta\,\left[\frac{d\left(\dfrac{1}{\cos\,\eta}\right)}{d\,\log\,\rho}\right] = E_0 + \varepsilon\, E_1 + \varepsilon^2\, E_2 + \dots$$

where

$$\begin{cases} E_0 = 4\,\sin^2\theta_i\,\cos^2\theta_i\,\left(\dfrac{d\,h_1}{d\,\log\,\rho}\right)^2 \\ E_1 = 8\,g_1 \dfrac{\cot\,\theta_i}{\sin\,2\theta_i}\,\dfrac{\tan^2\eta_i}{\cos^2\eta_i}\,\mathbf{C}_P^2\,\rho^2\,\sin^2\gamma + \\ \qquad 2\,\sin\,2\,\theta_i\,\dfrac{\tan\,\eta_i}{\cos\,\eta_i}\,\mathbf{C}_P\,\rho\,\sin\,\gamma\,\left(\dfrac{d\,h_1}{d\,\log\,\rho}\right) \end{cases} \tag{29}$$

Analogously

$$\rho^2\,\left[\left(\frac{1}{\cos\,\eta}\right)^2 - 1\right]\,\mathbf{C}_P^2\,\left(\frac{1}{\cos\,\eta}\right)^2\,\sin^2\gamma = M_0 + \varepsilon\, M_1 + \varepsilon^2\, M_2 + \dots$$

M_i being given by relations

$$\begin{cases} M_0 = \tan^2\eta_i\,\dfrac{1}{\cos^2\eta_i}\,\mathbf{C}_P^2\,\rho^2\,\sin^2\gamma \\ M_1 = \left[\,\tan^2\eta_i\,\dfrac{1}{\cos^2\eta_i}\,\left(2\,\mathbf{C}_P\,\dfrac{\partial\,\mathbf{C}_P}{\partial\,\mathbf{v}}\right) + \right. \\ \qquad \left.\left(\tan^2\eta_i\,\dfrac{2\,h_1}{\cos\,\eta_i} + \dfrac{1}{\cos^2\eta_i}\,\dfrac{2\,h_1}{\cos\,\eta_i}\right)\,\right]\,\rho^2\,\sin^2\gamma \end{cases} \tag{30}$$

Through identification we find

$$\begin{cases} h_1 = \dfrac{\sin\,\gamma}{\sin\,2\,\theta_i}\,\dfrac{\tan\,\eta_i}{\cos\,\eta_i}\,\displaystyle\int_{\rho_i}^{\rho}\mathbf{C}_P(v_i,\,\rho)\,d\,\rho \\ h_2 = \dfrac{\cos\,\eta_i\,\sin\,\gamma}{\tan\,\eta_i\,\sin^2 2\,\theta_i}\,\left[\,\sin\,\gamma\,(1 + 2\,\tan^2\eta_i)\,(1 + \tan^2\eta_i) + \right. \\ \qquad \left. 2\,\cot\,\theta_i\,\tan^2\eta_i\,(1 + \tan^2\eta_i)\,\cos\,\gamma\right] \end{cases} \tag{31}$$

At the same time one obtains

$$\left(\frac{1}{\cos\,\eta}\right)^2 \sin^2\theta\,\left[\frac{d\,(\tan\,\varphi)}{d\,\log\,\rho}\right)^2 = N_0 + \varepsilon\,N_1 + \varepsilon^2\,N_2 + \dots$$

where

$$\begin{cases} N_0 = \dfrac{1}{\cos^2\eta_i}\,\sin^2\theta_i\,\dfrac{d\,p_1}{d\,\log\,\rho} \\ N_1 = 2\,\dfrac{\sin^2\theta_i\,\cot^2\theta_i\,\tan^2\eta_i}{\cos^4\varphi_i}\,\left(\dfrac{d\,p_1}{d\,\log\,\rho}\right) + \\ \qquad 2\,\left(h_1\,\dfrac{\sin^2\theta_i}{\cos\,\eta_i} - g_1\,\dfrac{\cos\,\theta_i}{\cos^2\eta_i}\right)\,\dfrac{\cot^2\theta_i\,\sin^2\eta_i}{\cos^4\varphi_i} \end{cases} \tag{32}$$

The term of the second equation of the system under consideration is written

$$\cos^2\theta\,\left[\left(\frac{1}{\cos\,\eta}\right)^2 - 1\right]\,\left(1 + \tan^2\varphi\right) = T_0 + \varepsilon\,T_1 + \varepsilon^2\,T_2 + \dots$$

with T_i having the expressions

$$\begin{cases} T_0 = \cos^2\theta_i\,\tan^2\eta_i\,\dfrac{1}{\cos^4\varphi_i} \\ T_1 = \dfrac{2\,\sin^2\theta_i\,\cot^2\theta_i\,\tan^2\eta_i}{\cos^4\varphi_i}\,\left(\dfrac{d\,p_2}{d\,\log\,\rho}\right) + \\ \qquad \dfrac{2}{\cos^4\varphi_i}\,\left(h_1\,\dfrac{\cos^2\theta_i\,\sin^2\eta_i}{\cos\,\eta_i} - g_1\,\cos\,\theta_i\,\tan^2\eta_i\,\cot^2\theta_i\right) \end{cases} \tag{33}$$

By identifying and replacing the know expressions h_1, g_1 there will result the values

$$\begin{cases} p_1 = \dfrac{\cot^2\theta_i\,\sin^2\eta_i}{\cos^4\varphi_i}\,\log\,\dfrac{\rho}{\rho_i} \\ p_2 = \dfrac{1}{\sin\,\theta_i\,\sin\,2\theta_i}\,(\sin\,\theta_i\,\sin\,\gamma + \tan^2\eta_i\,\cos\,\gamma)\displaystyle\int_{\rho_i}^{\rho}\frac{d\,\rho}{\rho}\int_{\rho_i}^{\rho}\mathbf{C}_P(v_i,\,\rho)\,d\,\rho + \\ \dfrac{1}{2\,\sin^2\theta_i}\,\left[2\,\cos\,\theta_i\,\sin^2\eta_i\,\tan\,\varphi_i\,(1 + \tan\,\varphi_i) + \left(1 - \dfrac{1}{v_i^2}\right)\right]\,\left(\log\,\dfrac{\rho}{\rho_i}\right)^2 \end{cases} \tag{34}$$

From the last equation one deduces

$$\sin^2\theta\,\left(\frac{1}{\cos\,\eta}\right)^2\,\left[\frac{d\,\lambda}{d\,\log\,\rho}\right] = U_0 + \varepsilon\,U_1 + \varepsilon^2\,U_2 + \dots$$

where

$$\begin{cases} U_0 = \dfrac{1}{\cos^2\eta_i}\,\sin^2\theta_i\,\left(\dfrac{d\,g_1}{d\,\log\,\rho}\right)^2 \\ U_1 = \dfrac{\sin\,2\theta_i}{\cos\,\eta_i\,\cos\,\varphi_i}\,\left(\dfrac{d\,g_2}{d\,\log\,\rho}\right) + \dfrac{2}{\cos^2\varphi_i}\,\left(\cos^2\theta_i\,\cos\,\eta_i\,h_1 - \dfrac{\cos^3\eta_i}{\sin^2\theta_i}\,g_1\right) \end{cases} \tag{35}$$

The second term of the last equation becomes

$$\sin^2\theta\,(1 + \tan^2\varphi) = W_0 + \varepsilon\,W_1 + \varepsilon^2\,W_2 + \dots$$

with W_i being written as

$$\begin{cases} W_0 = \dfrac{\cos^2 \eta_i}{\cos^2 \varphi_i} \\ W_1 = 2\, \cos^2 \theta_i \, p_1 \, \tan\, \varphi_i + \dfrac{2\, g_1 \, \cos\, \theta_i}{\cos^2 \varphi_i} \end{cases} \tag{36}$$

By replacing in (35) and (36) the known terms one obtains

$$\begin{cases} q_1 = \cot^2 \theta_i \, \dfrac{\cos\, \eta_i}{\cos\, \varphi_i} \, \log\, \dfrac{\rho}{\rho_i} \\ q_2 = \dfrac{\cos\, \eta_i \, \cot\, \theta_i}{2} \left[\dfrac{1}{\sin^2 \theta_i \, \cos\, \varphi_i} \left(1 - \dfrac{1}{v_i^2} \right) + \cos\, \varphi_i \, \tan\, \varphi_i \, \cot^2 \theta_i \sin^2 \eta_i \right] \\ \left(\log\, \dfrac{\rho}{\rho_i} \right)^2 + \dfrac{1}{2} \left(\dfrac{\cos\, \gamma}{\sin^2 \theta_i \, \cos^2 \varphi_i} - \dfrac{\tan\, \eta_i}{2\, sin\, \theta_i \, \cos^2 \varphi_i} \right) \displaystyle\int_{\rho_i}^{\rho} \dfrac{d\, \rho}{\rho} \int_{\rho_i}^{\rho} \mathbf{C}_P(v_i,\, \rho)\, d\, \rho \end{cases} \tag{37}$$

The expressions obtained for $f_i,\, g_i,\, p_i,\, q_i,\; (i = 0,\, 1,\, 2,\, \ldots)$ enables us to obtain the solutions to the system of equations of spatial vehicles upon their re-entry into the atmosphere assuming the law of aerodynamic coefficient variation that as a given.

Although the integration method used is pretty fastidious, the results obtained are some of the best, ensuring a good approximation of the exact solution of the system of equations obtained through the numerical integration and having the advantage of some analytical expressions that yield the solution depending on the variation of the aerodynamic coefficients, as stated above.

By analyzing the system of equations, one notices that the terms of the system depend on the ratio $\dfrac{\mathbf{C}_P}{\mathbf{C}_R}$ which can be evaluated through the method of the small parameters used.

The results obtained are for the general three dimensional case, the solutions of the plane motion being obtained as a particular case.

References

1. W. T. Loh, *Entry mechanics and dynamics re-entry and planetary entry physics and technology*, Springer Verlag New York, 1968.
2. W. T. Loh, *Extension of second order theory of entry mechanics to oscillatory entry solutions*, AIAA Journal, pp. 1688-1691, september 1965.
3. M. Popescu, *Sur la stabilité de la penetration optimal des vehicules spatiales en atmosphère*, Rev. Rou. Math. Pures et Appl., Tome. **27**, No 10, pp. 1085-1089, 1982.
4. M. Popescu, *Periodic solutions for nonlinear differential system of equations with small parameter*, Nonlinear Analysis, vol. **52**, pp. 535-544, 2003.
5. M. Popescu, *Existence and orbital stability of periodic solutions for small parameter*, Nonlinear Analysis, vol. **33**, pp. 773-784, 1998.
6. M. Popescu, *Applications of canonical transformations in optimizing orbital transfer*, Journal of Guidance, Control and Dynamics, vol. **20**, No. 4, pp. 774-779, 1997.
7. M. Popescu, *Control of affine nonlinear systems with nilpotent structure in singular problems*, Journal of Optimization Theory and Application, vol. **124**, No. 2, pp. 455-466, 2005.
8. M. Popescu, *Optimal control for quasilinear systems with small parameters*, Journal of Applied Mathematics & Computing, vol. **27**, No. 1-2, pp. 393-409, 2008.