Existence and Uniqueness of the Optimal Control in Hilbert Space for a Class Linear Systems

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Abstract. One analyzes the existence and uniqueness of the optimal control in the class of exactly controllable linear systems. The cases under consideration refer to the minimization of time, energy and final variety in transfer problems.

The spaces of state variables $\mathbf{X}$ and control variables $\mathbf{U}$, respectively, are Hilbert spaces. The linear operator $\mathbf{T}(t)$ which defines the solution of the linear control system is a strong semi group. The present analysis resorts to the results of the theory of linear operators and functional analysis.

Keywords: existence and uniqueness, optimal control, controllable linear systems, linear operator.

Mathematics Subject Classifications (2000): 49J15 - 49N05.

1. Introduction

A particular importance should be assigned to the analysis of the control linear systems since they represent the mathematical model for numerous dynamic phenomena. One of the fundamental problems is the functional optimization that defines the performance index of the dynamic product. Thus, under differential and algebraic restrictions, one determines the control corresponding to functional extremisation under consideration. [5], [6].

Variational calculation offers methods that are difficult to investigate about the existence and uniqueness of optimal control.

The method of determining the field of extremals (sweep method) that analyzes the existence of the points conjugated across the optimal transfer trajectory (a sufficient optimum condition) is proof in this respect [8], [9] [12] [15]. Through their resulting applications, the minimization problems of time and energy represent an important objective in systems dynamics [4], [7], [10], [11], [13], [14].

For controllable systems, the recent results express the minimal energy through the controllability operator [1], [3], [14]. One also obtains the stability condition of the system for systems whose energy tends to zero in infinite time [2], [3]. By using linear operators in Hilbert spaces, in this study one analysis the existence and uniqueness of optimal control in transfer problems.

2. Minimum Time Control

2.1 Existence

2.1.1 Problem formulation

We consider the linear system ($\mathbf{\Sigma_{A,B}}$):

$$\frac{d\,\mathbf{x}}{d\,t} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbf{H} \tag{1}$$

where **H** is the Hilbert space whose internal product is $< \cdot, \cdot >$ and the norm is $\| \cdot \|$.

$\mathbf{A} : \mathbf{D}(\mathbf{A}) \subset \mathbf{H} \to \mathbf{H}$ is an unbounded operator on **H** which generates a strong semi group of operators on **H**, $(\mathbf{S}(t))_{t\geq 0} = (e^{t\mathbf{A}})_{t\geq 0}$.

$\mathbf{B} : \mathbf{U} \to \mathbf{H}$ is a bounded linear operator on another Hilbert space **U**, $\mathbf{B} \in \mathcal{L}(\mathbf{U}, \mathbf{H})$.

$\mathbf{u} : [0, \infty] \to \mathbf{U}$ is a square integrable function representing the system control (1)

For any control function **u** there exists a solution for (1) given by

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0 + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{B}\,\mathbf{u}(s)\,d\,s\,, \qquad t \geq 0 \tag{2}$$

The problem of control is:

Given $t_0 \in \mathbf{I}$, $\mathbf{x}(t_0)$, $\mathbf{z} \in \mathbf{X}$ and the constant $M > 0$, let us determine $\mathbf{u} \in \mathbf{U}$ so that $\|\mathbf{u}\| \leq M$ and there are satisfied the conditions

$$\begin{aligned} &(i) \quad \mathbf{x}(t_1) = \mathbf{z} \\ &(ii) \quad t_1 = \inf\{\, t \,|\, \mathbf{x}(t) = \mathbf{z}\,\} \end{aligned} \tag{3}$$

Spaces **X** and **U** are Hilbert spaces.

Theorem 1.

The control of optimal time exists for the problem formulated

Proof

Let $\{t_n\}$ a decreasing monotonous series thus that $t_n \to t_1$ and $\{\mathbf{u}_n(\tau)\}$ the elements $\mathbf{u}_n = \mathbf{u}(t_n) \in \mathbf{U}$. We have

$$\mathbf{x}(t_n) = \mathbf{S}(t_n)\mathbf{x}(t_0) + \int_{t_0}^{t_n} \mathbf{S}(t_n - \tau)\,\mathbf{B}\,\mathbf{u}_n(\tau)\,d\,\tau \tag{4}$$

which can also be written

$$\mathbf{x}(t_n) = \mathbf{S}(t_n)\mathbf{x}(t_0) + \int_{t_0}^{t_1} \mathbf{S}(t_n - \tau)\,\mathbf{B}\,\mathbf{u}_n(\tau)\,d\,\tau + \int_{t_1}^{t_n} \mathbf{S}(t_n - \tau)\,\mathbf{B}\,\mathbf{u}_n(\tau)\,d\,\tau \tag{5}$$

$\mathbf{S}(t)$ is a continuous linear operator, therefore bordet. It follows that we have:

$$\mathbf{S}(t_n)\mathbf{x}(t_0) \to \mathbf{S}(t_1)\mathbf{x}(t_0) \tag{6}$$

$$\int_{t_1}^{t_n} \mathbf{S}(t_n - \tau)\,\mathbf{B}\,\mathbf{u}_n(\tau)\,d\,\tau \to 0 \tag{7}$$

We consider the multitude of admissible controls as a closed convex domain and weakly compact defined by

$$\mathbf{\Omega} = \{\, \mathbf{u} \in \mathbf{U} \,|\, \|\mathbf{u}\| \leq M\,\} \tag{8}$$

As $\mathbf{\Omega} \subset \mathbf{U}$ is weakly compact, any sequence $(\mathbf{u}) \in \mathbf{\Omega}$ allows a weakly convergent sequence at $\mathbf{u}_1 \in \mathbf{\Omega}$.

Therefore for series $(\mathbf{u}) \in \mathbf{\Omega}$ to exist one subseries $(\mathbf{u}_i)$ weakly convergent at $\mathbf{u}_1 \in \mathbf{\Omega}$. Thus, for $\mathbf{u}_1 \in \mathbf{\Omega}$ and any $\mathbf{u}^* \in \mathbf{\Omega}^*$ (the dual of $\mathbf{\Omega}$), we have

$$\lim_{i\to\infty} \langle \mathbf{u}_i,\, \mathbf{u}^* \rangle = \langle \mathbf{u}_1,\, \mathbf{u}^* \rangle \tag{9}$$

Let $\mathbf{u}_1 \in \mathbf{\Omega}$, which implies

$$\| \mathbf{u}_1 \| \leq M \tag{10}$$

We also have

$$\begin{gathered} \mathbf{B}^* \;:\; \mathbf{X} \;\to\; \mathbf{U} \\ \mathbf{S}^*(t_n - \tau)\,\mathbf{x} \in \mathbf{U} \end{gathered} \tag{11}$$

It follows

$$\mathbf{B}^*\, \mathbf{S}^*(t_n - \tau)^*\, \mathbf{x} \in \mathbf{U} \tag{12}$$

For every $\mathbf{x} \in \mathbf{X}$ one evaluates the difference

$$\mathbf{F} = \left\langle \int_{t_0}^{t_1} \mathbf{S}(t_n - t_0)\, \mathbf{B}\, \mathbf{u}_n(\tau)\, d\,\tau,\, \mathbf{x} \right\rangle - \left\langle \int_{t_0}^{t_1} \mathbf{S}(t_1 - t_0)\, \mathbf{B}\, \mathbf{u}_1(\tau)\, d\,\tau,\, \mathbf{x} \right\rangle \tag{13}$$

By writing

$$\int_{t_0}^{t_1} \mathbf{S}(t_k - t_0)\, \mathbf{B}\, d\,\tau = \mathcal{L}_k \qquad k = 1,\, n \tag{14}$$

the expression (13) becomes

$$\langle \mathcal{L}_n\, \mathbf{u}_n,\, \mathbf{x} \rangle \,-\, \langle \mathcal{L}_1\, \mathbf{u}_1,\, \mathbf{x} \rangle \,=\, \langle \mathbf{u}_n,\, \mathcal{L}_n^*\, \mathbf{x} \rangle \,-\, \langle \mathbf{u}_1,\, \mathcal{L}_1^*\, \mathbf{x} \rangle \tag{15}$$

or

$$\mathbf{F} = \langle \mathbf{u}_n,\, \mathcal{L}_1^*\, \mathbf{x} \rangle \,-\, \langle \mathbf{u}_1,\, \mathcal{L}_1^*\, \mathbf{x} \rangle \,-\, \langle \mathbf{u}_n,\, \mathcal{L}_1^*\, \mathbf{x} \rangle \,+\, \langle \mathbf{u}_n,\, \mathcal{L}_n^*\, \mathbf{x} \rangle \tag{16}$$

From where

$$\begin{gathered} \mathbf{F} = \langle \mathbf{u}_n - \mathbf{u}_1,\, \mathcal{L}_1^*\, \mathbf{x} \rangle \,+\, \langle \mathbf{u}_n,\, (\, \mathcal{L}_n^* - \mathcal{L}_1^*\,)\, \mathbf{x} \rangle \,= \\ \int_{t_0}^{t_1} \langle\, \mathbf{u}_n(\tau) - \mathbf{u}_1(\tau),\, \mathbf{B}^*\, \mathbf{S}(t_1 - \tau)^*\, \mathbf{x} \,\rangle\, d\,\tau + \\ \int_{t_0}^{t_1} \langle\, \mathbf{u}_n(\tau),\, \mathbf{B}^*\, \mathbf{S}(t_n - \tau)^*\, \mathbf{x} - \mathbf{B}^*\, \mathbf{S}(t_1 - \tau)^*\, \mathbf{x} \,\rangle\, d\,\tau \end{gathered} \tag{17}$$

By using the properties of operator $\mathbf{S}(t)$ one obtains

$$\begin{gathered} \mathbf{F} = \int_{t_0}^{t_1} \langle\, (\mathbf{u}_n(\tau) - \mathbf{u}_1(\tau))\,,\, \mathbf{B}^*\, \mathbf{S}(t_1 - \tau)^*\, \mathbf{x} \,\rangle\, d\,\tau + \\ \int_{t_0}^{t_1} \langle\, \mathbf{u}_n(\tau),\, \mathbf{B}^*\, \mathbf{S}(t_1 - \tau)^*\, [\, \mathbf{S}t_n - t_1)^*\, \mathbf{x} - \mathbf{x}] \,\rangle\, d\,\tau \end{gathered} \tag{18}$$

Because series $\mathbf{u}_n$ converges weakly at $\mathbf{u}_1$, the first term of expression (18) tends to zero at $n \to \infty$ (see (9)).

$\mathbf{S}(t_1 - \tau)^*$ is a semigroup continuous hence bounded, so that there exists the constant $K > 0$ for which

$$\left\| \mathbf{B}^* \, \mathbf{S}(t_1 - \tau)^* \left[\mathbf{S}(t_n - t_1)^* \, \mathbf{x} - \mathbf{x} \right] \right\| \leq K \, \| \mathbf{S}(t_n - t_1)^* \, \mathbf{x} - \mathbf{x} \| \tag{19}$$

It follows that the second term from (18) tends to zero at $n \to \infty$.

The sequence $\{ \mathbf{x}(t_n) \} \in \mathbf{X}$ is weakly convergent at $\mathbf{x}(t_1) = \mathbf{z} \in \mathbf{X}$ if for any $\mathbf{x} \in \mathbf{X}^* = \mathbf{X}$, we have

$$\lim_{n \to \infty} \langle \, \mathbf{x}(t_n), \, \mathbf{x} \, \rangle = \langle \, \mathbf{x}(t_1), \, \mathbf{x} \, \rangle \tag{20}$$

which becomes

$$\langle \, \mathbf{z}, \, \mathbf{x} \, \rangle = \left\langle \mathbf{S}(t_1) \, \mathbf{x}(t_0) + \int_{t_0}^{t_1} \mathbf{S}(t_1 - \tau) \, \mathbf{B} \, \mathbf{u}_1(\tau) \, d\,\tau, \, \mathbf{x} \right\rangle \tag{21}$$

From (21) one gets which becomes

$$\mathbf{S}(t_1) \, \mathbf{x}(t_0) + \int_{t_0}^{t_1} \mathbf{S}(t_1 - \tau) \, \mathbf{B} \, \mathbf{u}_1(\tau) \, d\,\tau = \mathbf{z} \tag{22}$$

and hence $\mathbf{u}_1$ is the optimal control. □

An important result which is going to be used in demonstrating uniqueness belongs to A. Friedman:

Theorem 2 (bang-bang).

We assume that the multitude $\mathbf{\Omega}$ is convex in the neighborhood of the origin and $\mathbf{u}(t)$ is the control of optimal time in the problem formulated by (2.1.1)

$\Longrightarrow \mathbf{u}(t) \in \mathbf{\Omega}$ for almost all $t \in [t_0, \, t_1]$.

2.2 Uniqueness of time - optimal control

2.2.1 Rotund space

Let β the unity sphere in the Banach space $\mathbf{U}$ and let $\partial \beta$ its boundary.

Space $\mathbf{U}$ is rotund if the following equivalent conditions are satisfied.

(i) $\| \mathbf{x}_1 + \mathbf{x}_2 \| = \| \mathbf{x}_1 \| + \| \mathbf{x}_2 \| \implies$ a scalar exists $\lambda \neq 0$ so that $\mathbf{x}_2 = \lambda \, \mathbf{x}_1$

(ii) Each convex sub multitude $\mathbf{K} \subset \mathbf{U}$ has at least one element that satisfies

$$\| \mathbf{u} \| \leq \| \mathbf{z} \| \,, \quad \mathbf{u} \in \mathbf{K}, \quad \mathbf{z} \in \mathbf{K}$$

(iii) For any bounded linear function $\mathbf{f}$ on $\mathbf{U}$ there is at least an $\mathbf{x} \in \beta$ so that

$$\langle \, \mathbf{x}, \, \mathbf{f} \, \rangle = \mathbf{f}(\mathbf{x}) = \| \mathbf{f} \|$$

(iv) Each $\mathbf{x} \in \partial \beta$ is a point of extreme of a β.

Examples of rotund space:

1. Hilbert spaces.
2. Spaces $\mathbf{l}^p$, $\mathbf{L}^p$, $1 < p < \infty$.
3. If the Banach spaces $\mathbf{U}_1, \, \mathbf{U}_2, \, \ldots, \, \mathbf{U}_n$ are rotund $\Longrightarrow$ space $\mathbf{U}_1 \times \mathbf{U}_2 \times \ldots \times \mathbf{U}_n$ is rotund.
4. Uniform convex spaces are rotund, only the reciprocal is not true.

2.2.2 Uniqueness

We assume that $\mathbf{u}_1$ and $\mathbf{u}_2$ are optimal, $\mathbf{u}_i \in \mathbf{U}$, $i = 1,\ 2$.
Thus there are satisfied the equations

$$\mathbf{S}(t_1)\,\mathbf{x}(t_0) + \int_{t_0}^{t_1} \mathbf{S}(t_1 - \tau)\,\mathbf{B}\,\mathbf{u}_1(\tau)\,d\,\tau = \mathbf{z} \tag{23}$$

$$\mathbf{S}(t_1)\,\mathbf{x}(t_0) + \int_{t_0}^{t_1} \mathbf{S}(t_1 - \tau)\,\mathbf{B}\,\mathbf{u}_2(\tau)\,d\,\tau = \mathbf{z} \tag{24}$$

respectively.

By adding equations (23) and (24) one obtains

$$\mathbf{S}(t_1)\,\mathbf{x}(t_0) + \int_{t_0}^{t_1} \mathbf{S}(t_1 - \tau)\,\mathbf{B}\,\cdot\,\frac{1}{2}\,\left(\mathbf{u}_1(\tau) + \mathbf{u}_2(\tau)\right)\,d\,\tau = \mathbf{z} \tag{25}$$

It follows that $\mathbf{u}_1(\tau)$, $\mathbf{u}_2(\tau)$, $\frac{1}{2}\,(\mathbf{u}_1(\tau) + \mathbf{u}_2(\tau))$ are optimal.
By using Theorem 2 we have

$$\mathbf{u}_1(\tau),\ \mathbf{u}_2(\tau),\ \frac{1}{2}\,(\mathbf{u}_1(\tau) + \mathbf{u}_2(\tau)) \in \partial\beta \implies \|\,\mathbf{u}_k\,\| = 1,\ \frac{1}{2}\,\|\,\mathbf{u}_1 + \mathbf{u}_2\,\| = 1$$

Since condition (i) is satisfied, $\mathbf{U}$ is a rotund space and $\mathbf{u}_1 = \lambda\,\mathbf{u}_2$.
Thus

$$\|\,\mathbf{u}_1\,\| + \|\,\mathbf{u}_2\,\| = (\,1 + |\,\lambda\,|\,)\,\|\,\mathbf{u}_2\,\| = 2 \tag{26}$$

from where

$$|\,\lambda\,| = 1 \implies \mathbf{u}_1 = \mathbf{u}_2 \tag{27}$$

Uniqueness has been demonstrated. □

3. Minimum Energy Problem

3.1 Problem formulation

Let $\Sigma_{\mathbf{A},\,\mathbf{B}}$ (see (1)) be a controllable system with finite dimensional state space $\mathbf{X}$.
Let $\mathbf{I} = [0,\ t_1]$, $\mathbf{x}(0) = \mathbf{a} \in \mathbf{X}$, $\mathbf{b} \in \mathbf{X}$ be given. Let $\mathbf{U}$ be Hilbert space $\mathbf{L}^2(\mathbf{I};\ \mathbf{U})$.
The minimum norm control problem is then to determine $\mathbf{u}(t) \in \mathbf{U}$ such that for some $t_1 \in \mathbf{I}$.
(i) $\mathbf{x}(t_1) = \mathbf{b}$,
(ii) $\|\,\mathbf{u}\,\|$ is minimized on $[0,\ t_1]$ where $\|\cdot\|$ represents the norm on $\mathbf{L}^2(\mathbf{I};\ \mathbf{U})$.
Let $t_1 > 0$ fixed and we consider the linear operator

$$\mathcal{L}_{t_1}\ :\ \mathbf{L}^2[0,\ t_1;\ \mathbf{U}]\ \to\ \mathbf{H}_1 \tag{28}$$

defined by

$$\mathcal{L}_{t_1}\,\mathbf{u} = \int_0^{t_1} \mathbf{S}(t_1 - s)\,\mathbf{B}\,\mathbf{u}(s)\,d\,s \tag{29}$$

Thus we have

$$\mathbf{x}(t_1) = \mathbf{S}(t_1)\,\mathbf{a} + \mathcal{L}_{t_1}\,\mathbf{u} = \mathbf{b} \tag{30}$$

Theorem 3.

Let $\mathcal{L}_t$ be a linear mapping between a Hilbert space $\mathbf{U}$ and a finite dimensional space $\mathbf{H}$, $\mathcal{L}_t : \mathbf{U} \to \mathbf{H}$.

$\implies$ Exists a finite dimensional space $\mathbf{M} \subset \mathbf{U}$ such that $\mathcal{L}_t$ restricted to $\mathbf{M}$, $\mathcal{L}_t^{\mathbf{M}}$ is an injective mapping.

Proof

Let $\{ e_i \}$ $i = 1, \dots, n$, be a basis for range $\mathcal{L}_t \subset \mathbf{H}$.

Given any $\mathbf{u} \in \mathbf{U}$, $\mathcal{L}\,\mathbf{u}$ can be expressed

$$\mathcal{L}_t\,\mathbf{u} = \sum_{i=1}^{n} \alpha_i\, e_i \tag{31}$$

where each of the α_i can be expressed

$$\alpha_i = \langle\, \mathbf{f}_i, \mathbf{u}\,\rangle \qquad \mathbf{f}_i \in \mathbf{U}^* = \mathbf{U} \tag{32}$$

Then

$$\mathcal{L}_t\,\mathbf{u} = \sum_{i=1}^{n} \langle\, \mathbf{f}_i, \mathbf{u}\,\rangle\, e_i \tag{33}$$

The $\mathbf{f}_i$ are linearly independent and generate an n dimensional subspace say $\mathbf{M} \subset \mathbf{U}$.

From the properties of Hilbert space (projection theorem), $\mathbf{U} = \mathbf{M} \oplus \mathbf{M}^\perp$.

Let $\mathbf{u} \in \mathbf{M}^\perp$ then $\langle\, \mathbf{f}_i, \mathbf{u}\,\rangle = \alpha_i = 0$.

Thus

$$\mathbf{M}^\perp \subset \{\, \mathbf{u} \in \mathbf{U} \mid \mathcal{L}_t\,\mathbf{u} = 0\,\} = \ker\, \mathcal{L}_t \tag{34}$$

Let $\mathbf{u} \in \ker\, \mathcal{L}_t$, then

$$\mathcal{L}_t\,\mathbf{u} = 0 = \sum_{i=1}^{n} \langle\, \mathbf{f}_i, \mathbf{u}\,\rangle\, e_i \tag{35}$$

Since the e_i are independent, each $\langle\, \mathbf{f}_i, \mathbf{u}\,\rangle = 0$, $i = 1, \dots, , n$ so that $\mathbf{M}^\perp = \ker\, \mathcal{L}_t$.

$\mathcal{L}_t$ maps $\mathbf{M}$ bijectively to range $\mathcal{L}_t$ and hence $\mathcal{L}_t^{\mathbf{M}}$ is injective mapping into $\mathbf{H}$.

Let

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2\,,\;\; \mathbf{u}_1 \in \mathbf{M}\,,\;\; \mathbf{u}_2 \in \mathbf{M}^\perp\,,\;\; \mathbf{u}_2 \neq 0 \tag{36}$$

Because $\mathcal{L}_t\,\mathbf{u}_2 = 0$,

$$\mathcal{L}_t\,\mathbf{u} = \mathcal{L}_t^{\mathbf{M}}\,\mathbf{u}_1 = \mathbf{x} \in \mathbf{H} \tag{37}$$

Since $\langle\, \mathbf{u}_1, \mathbf{u}_2\,\rangle = 0$, we have

$$\|\,\mathbf{u}\,\|^2 = \langle\, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_2\,\rangle = \|\,\mathbf{u}_1\,\|^2 + \|\,\mathbf{u}_2\,\|^2 \tag{38}$$

Then a unique minimum norm $\left(\mathcal{L}_t^{\mathbf{M}}\right)^{-1}\,\mathbf{x}$ exists, and $\mathbf{u}_1 \in \mathcal{L}_t^{\mathbf{M}}\,\mathbf{x}$ is the minimum norm element satisfying $\mathcal{L}\,\mathbf{u} = \mathbf{x}$.

Remark 1

The unique solution $\mathbf{u}(t)$ of the equation $\mathcal{L}\,\mathbf{u} = \mathbf{x}$, with minimum norm control is the projection of $\mathbf{u}(t)$ onto the closed subspace $\mathbf{M} = (\,\mathbf{f}_1, \dots, \mathbf{f}_n\,)$.

Hence from (30) exists a control $\mathbf{u}(\cdot) \in \mathbf{L}^2[0,\, t_1;\, \mathbf{H}_1]$ transferring a to $\mathbf{b}$ in time t_1 if and only $(\mathbf{b} - \mathbf{S}(t_1)\,\mathbf{a}) \in Im\,\mathcal{L}_{t_1}$.

The control which achieves this and minimizes the functional $\mathbf{E}_{t_1} = \int_0^{t_1} \|\mathbf{u}(s)\|^2\, d\,s$ called the energy functional is

$$\mathbf{u} = \mathcal{L}_{t_1}^{-1}\,(\mathbf{b} - \mathbf{S}(t_1)\,\mathbf{a}) \tag{39}$$

Define the linear operator

$$\mathbf{Q}_t = \int_0^t \mathbf{S}(r)\,\mathbf{B}\,\mathbf{B}^*\mathbf{S}(r)\,d\,r \qquad t \geq 0 \tag{40}$$

We have the following results (see [1], [13], [14]):

Proposition 1.

(i) The function $\mathbf{Q}_t$, $t \geq 0$, is the unique solution of the equation

$$\frac{d}{d\,t}\,\langle\,\mathbf{Q}_t\,\mathbf{x},\, \mathbf{x}\,\rangle = 2\,\langle\,\mathbf{Q}_t^*\,\mathbf{x},\, \mathbf{x}\,\rangle + \|\,\mathbf{B}^*\,\mathbf{x}\,\|^2 \qquad \mathbf{x} \in D(\mathbf{A}^*),\ \mathbf{Q}_0 = \mathbf{I} \tag{41}$$

where

$$D(\mathbf{A}^*) = \left\{\mathbf{y} \in \mathbf{H}\,|\,\exists\, C \in \mathbf{R}^+,\ \left|\langle\,\mathbf{A}\,\mathbf{x},\, \mathbf{y}\,\rangle_{\mathbf{H}}\right| \leq C\ \|\,\mathbf{x}\,\|_{\mathbf{H}}\,, \forall\, \mathbf{x} \in D(\mathbf{A})\right\} \tag{42}$$

(ii) If $\mathbf{A}$ generates a stable semigroup then

$$\lim_{t \to \infty}\,\mathbf{Q}_t = \mathbf{Q} \tag{43}$$

exists and is the unique solution of the equation

$$2\,\langle\,\mathbf{Q}\,\mathbf{A}^*\,\mathbf{x},\, \mathbf{x}\,\rangle + \|\,\mathbf{B}^*\,\mathbf{x}\,\|^2 = 0 \qquad \mathbf{x} \in D(\mathbf{A}^*) \tag{44}$$

The proof of Proposition 1 is given by the author of this study in [13], [14].

The following theorem gives general results for the functionals $\mathbf{E}_{t_1}(\mathbf{a},\, \mathbf{b})$, the minimal energy of transferring $\mathbf{a}$ to $\mathbf{b}$ in time t_1, and $\mathbf{E}_\infty(0,\, \mathbf{b})$, $\mathbf{a}$, $\mathbf{b} \in \mathbf{H}$, $t_1 > 0$ (see [1]).

Theorem 4.

(i) For arbitrary $t_1 > 0$ and $\mathbf{a}$, $\mathbf{b} \in \mathbf{H}$

$$\mathbf{E}_{t_1}(\mathbf{a},\, \mathbf{b}) = \left\|\left(\mathbf{Q}_{t_1}^{1/2}\right)^{-1}\,(\,\mathbf{S}(t_1)\,\mathbf{a} - \mathbf{b}\,)\right\|^2 \tag{45}$$

(ii) If $\mathbf{S}(t)$ is stable and the system $(\,\Sigma_{\mathbf{A},\mathbf{B}}\,)$ is null controllable in time $t_0 > 0$, then

$$\mathbf{E}_\infty(0,\, \mathbf{b}) = \left\|\left(\mathbf{Q}^{1/2}\right)^{-1}\,\mathbf{b}\right\|^2 \quad \mathbf{b} \in \mathbf{H} \tag{46}$$

(iii) Moreover, there exists $C_{t_0} > 0$, such that

$$\left\|\left(\mathbf{Q}^{1/2}\right)^{-1}\,\mathbf{b}\right\|^2 \leq \mathbf{E}_{t_1}(0,\, \mathbf{b}) \leq C_{t_0}\,\left\|\left(\mathbf{Q}^{1/2}\right)^{-1}\,\mathbf{b}\right\|^2 \quad \mathbf{b} \in \mathbf{H} \quad t_1 \geq t_0 \tag{47}$$

4. Numerical Methods for Minimal Norm Control

4.1 Presentation of the Numerical Methods

Let $\Sigma_{\mathbf{A},\mathbf{B}}$ be a dynamic system with $\mathbf{U} = \mathbf{R}^m$, $\mathbf{X} = \mathbf{R}^n$, $\mathbf{I} = \mathbf{R}^1$.
Given $\mathbf{x}_0 = 0$, t_0, t_1, $\mathbf{b}$ determine $\mathbf{u}(t) \in \mathbf{U}$ such that
(i) $\mathbf{x}(t_1) = \mathbf{b}$,
(ii) $\left\| \mathbf{u}_p \right\|$ is minimized $p \in [1, \infty)$.
Now

$$\mathbf{e} = \mathbf{b} = \int_{t_0}^{t_1} \mathbf{S}(t_1 - \tau)\,\mathbf{B}\,\mathbf{u}(\tau)\,d\tau = \int_{t_0}^{t_1} \mathbf{\Phi}(t_1 - \tau)\,\mathbf{u}(\tau)\,d\tau \tag{48}$$

$\mathbf{u}(t)$ must be chosen on the interval $[t_0, t_1]$ to make equation (48) true, $\mathbf{e}$ is a vector.

Consider its ith component e_i

$$e_i = \int_{t_0}^{t_1} \varphi_i(t_1 - \tau)\,\mathbf{u}(\tau)\,d\tau = \langle \varphi_i, \mathbf{u} \rangle = \mathbf{f}(\varphi_i) \tag{49}$$

where φ_i is the row of the matrix $\mathbf{\Phi}$ and $\mathbf{f}$ is the unique functional corresponding to the inner product (Riesz reprezentation theorem).

Let λ_i be an arbitrary scalar then

$$\lambda_i\, e_i = \lambda_i\, \mathbf{f}(\varphi_i) = \mathbf{f}(\lambda_i\, \varphi_i) \tag{50}$$

Because $\mathbf{R}^n$ is Hilbert space, the inner product

$$\sum_{i=1}^{n} \lambda_i\, e_i = \sum_{i=1}^{n} \mathbf{f}(\lambda_i\, \varphi_i) = \langle \lambda, \mathbf{e} \rangle \tag{51}$$

λ being a vector with arbitrary elements $\lambda_1, \ldots, \lambda_n$.

If equation (51) is true for at least n different linearly independent then equation (48) will also is true.

We have

$$\left| \mathbf{f}\left(\sum_{i=1}^{n} \lambda_i\, \varphi_i \right) \right| = \left| \left\langle \sum_{i=1}^{n} \lambda_i\, \varphi_i, \mathbf{u} \right\rangle \right| \tag{52}$$

By Hölders inequality,

$$\left| \left\langle \sum_{i=1}^{n} \lambda_i\, \varphi_i, \mathbf{u} \right\rangle \right| \le \| \mathbf{u} \|_p \left\| \sum_{i=1}^{n} \lambda_i\, \varphi_i \right\|_q , \qquad \left(\frac{1}{p} + \frac{1}{q} = 1 \right) \tag{53}$$

From equation (51), (53),

$$\| \mathbf{u} \|_p \ge \frac{\left| \mathbf{f}\left(\sum_{i=1}^{n} \lambda_i\, \varphi_i \right) \right|}{\left\| \sum_{i=1}^{n} \lambda_i\, \varphi_i \right\|_q} = \frac{\sum_{i=1}^{n} \lambda_i\, e_i}{\left\| \sum_{i=1}^{n} \lambda_i\, \varphi_i \right\|_q} = \frac{\langle \lambda, \mathbf{e} \rangle}{\left\| \sum_{i=1}^{n} \lambda_i\, \varphi_i \right\|_q} \tag{54}$$

Every control driving the system to the point **b** must satisfy equation (54) while for optimal (minimum norm) control. Equation (54) must be satisfied with equality (Theorem 2).

Numerically, are must search for **a** n vector λ such that the right side of equation (54) tanks on its maximum.

4.2 A Simple Example

We consider a single output linear dynamic system $\Sigma_{\mathbf{A}\,\mathbf{B}}$ where

$$\mathbf{U} = \mathbf{R}^1\,, \quad \mathbf{X} = \mathbf{R}^1\,, \quad \mathbf{I} = \mathbf{R}^1$$

If condition (i) is t_0 be satisfied then we must have.

Because $t_1 \in \mathbf{I}$ fixed, then $\mathbf{S}(t_1 - \tau)\,\mathbf{B} = \mathbf{\Phi}(\tau)$

$$\mathbf{e} = \mathbf{b} = \int_{t_0}^{t_1} \mathbf{\Phi}(\tau)\,\mathbf{u}(\tau)\,d\tau \tag{55}$$

Therefore

$$|\,\mathbf{e}\,| = \left| \int_{t_0}^{t_1} \mathbf{\Phi}(\tau)\,\mathbf{u}(\tau)\,d\tau \right| \le \int_{t_0}^{t_1} |\,\mathbf{\Phi}(\tau)\,\mathbf{u}(\tau)\,|\;d\tau \tag{56}$$

and from the Hölders inequality

$$\int_{t_0}^{t_1} |\,\mathbf{\Phi}(\tau)\,\mathbf{u}(\tau)\,|\;d\tau \le \|\,\mathbf{\Phi}\,\|\;\|\,\mathbf{u}\,\| \tag{57}$$

where it has been assumed that $\mathbf{\Phi}(t) \in \mathbf{L}^2(\mathbf{I})$.

The minimum norm control, if it exists, will satisfy

$$\|\,\mathbf{u}\,\| = \frac{|\,\mathbf{e}\,|}{\|\,\mathbf{\Phi}\,\|} \tag{58}$$

From (56), and from (57) we have

$$\operatorname{sign}(\mathbf{u}(t)) = \operatorname{sign}(\mathbf{\Phi}(t))\,, \qquad \forall\, t \in [t_0,\, t_1] \tag{59}$$

respectively

$$|\,\mathbf{\Phi}(t)\,|^q = h\,|\,\mathbf{u}(t)\,|^p\,, \qquad \forall\, t \in [t_0,\, t_1]\,,\ h \text{ arbitrary constant} \tag{60}$$

The control **u** satisfies the relation

$$|\,\mathbf{u}(t)\,| = h^{-1/p}\,|\,\mathbf{\Phi}(t)\,|^{q/p} = k\,|\,\mathbf{\Phi}(t)\,|^{q/p} \tag{61}$$

or

$$\mathbf{u}(t)\operatorname{sign}(\mathbf{u}(t)) = k\,|\,\mathbf{\Phi}(t)\,|^{q/p} \tag{62}$$

From condition (59),

$$\mathbf{u}(t) = k\operatorname{sign}(\mathbf{\Phi}(t))\,|\,\mathbf{\Phi}(t)\,|^{q/p} \tag{63}$$

By substituting equation (63) into equation 955), the constant k can be determined.

4.2.1 Application in particular case $p = q = 2$

In this case the equation (55) represent the inner product in $\mathbf{L}^2(t_0,\ t_1;\ \mathbf{U})$,

$$\mathbf{b} = \langle\, \boldsymbol{\Phi},\ \mathbf{u}\,\rangle \tag{64}$$

The relation (63) becomes

$$\mathbf{u}(t) = k\,\mathrm{sign}\,(\boldsymbol{\Phi}(t))\ |\,\boldsymbol{\Phi}(t)\,| \tag{65}$$

Then

$$\mathbf{b} = \int_{t_0}^{t_1} k\,\boldsymbol{\Phi}^2(\tau)\,d\,\tau = k\ \|\,\boldsymbol{\Phi}\,\|^2 \tag{66}$$

Thus

$$k = \frac{\mathbf{b}}{\|\,\boldsymbol{\Phi}\,\|^2} \tag{67}$$

and the optimal control is given by

$$\mathbf{u}(t) = \frac{\mathbf{b}\,\boldsymbol{\Phi}(t)}{\|\,\boldsymbol{\Phi}\,\|^2} \tag{68}$$

5. Exact Controllability

5.1 Adjoint System

Let $\mathbf{S}(t)$, $t \in [0,\ t_1]$, the fundamental solution of a homogeneous system associated to the linear control system $\Sigma_{\mathbf{A},\,\mathbf{B}}$ (see (1))

Thus we have

$$\frac{d\,\mathbf{S}(t)}{d\,t} = \mathbf{A}\,\mathbf{S}(t) \qquad\qquad \mathbf{S}(0) = \mathbf{I}\,;\ \ t \in [0,\ t_1] \tag{69}$$

From condition

$$\frac{d}{d\,t}\,\Big(\mathbf{S}(t)\,\mathbf{S}^{-1}(t)\Big) = \frac{d\,\mathbf{I}}{d\,t} = 0 \tag{70}$$

one obtains

$$\frac{d}{d\,t}\,\mathbf{S}^{-1}(t) = -\mathbf{S}^{-1}(t)\,\mathbf{A} \tag{71}$$

wherefrom

$$\Big(\mathbf{S}^{-1}(t)\Big)^* = -\mathbf{A}^*\,\Big(\mathbf{S}^{-1}(t)\Big)^* \tag{72}$$

Since

$$\Big(\mathbf{S}^{-1}(t)\Big)^* = (\,\mathbf{S}^*(t)\,)^{-1} \tag{73}$$

The system becomes

$$\frac{d}{d\,t}\,\Big[\,(\,\mathbf{S}^*(t)\,)^{-1}\,\Big] = -\mathbf{A}^*\ (\,\mathbf{S}^*(t)\,)^{-1}\ , \qquad t \in [0,\ t_1] \tag{74}$$

It follows that $(\,\mathbf{S}^*(t)\,)^{-1}$, $t \in [0,\ t_1]$ is the fundamental solution for the adjoint system to (69).

5.2 A Class of Linear Controllable Systems

We consider the class of linear control systems in vectorial form

$$\frac{d\,\mathbf{x}}{d\,t} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} + \mathbf{C}\,, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{75}$$

Let $\mathbf{u}_0$ any internal point of the closed bounded convex control domain $\mathbf{U}$. This domain contains the space $\mathbf{E}_m$ of variables $\mathbf{u} = (u_1, \ldots, u_n)$.

We take

$$\bar{\mathbf{u}} = \mathbf{u} - \mathbf{u}^\circ \tag{76}$$

With this transformation, system (75) becomes

$$\frac{d\,\mathbf{x}}{d\,t} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\bar{\mathbf{u}} + (\,\mathbf{B}\,\mathbf{u}^\circ + \mathbf{C}\,) \tag{77}$$

Thus, one transfers the origin of the coordinates of space $\mathbf{E}_m$ in $\mathbf{u}^\circ$.

At the same time, the origin of the coordinates of space $\mathbf{E}_m$ is an internal point inside domain $\mathbf{U}$.

We write $\mathbf{x}^\circ(t)$ for the solution of system (75) which corresponds to the control $\mathbf{u} \equiv 0$.

This control is admissible as the origin of the coordinates belongs to the domain $\mathbf{U}$ and satisfies the initial condition $\mathbf{x}^\circ(0) = \mathbf{x}_0$.

The result is

$$\frac{d\,\mathbf{x}^\circ(t)}{d\,t} = \mathbf{A}\,\mathbf{x}^\circ(t) + \mathbf{C} \tag{78}$$

Let $\mathbf{u}_1(t)$, $0 \le t \le t_1$, any control and $\mathbf{x}_1(t)$ the trajectory corresponding to system (75).

We have

$$\frac{d\,\mathbf{x}_1(t)}{d\,t} = \mathbf{A}\,\mathbf{x}_1(t) + \mathbf{B}\,\mathbf{u}_1(t) + \mathbf{C}\,, \quad \mathbf{x}_1(0) = \mathbf{x}_0 \tag{79}$$

We take

$$\bar{\mathbf{x}}(t) = \mathbf{x}_1(t) - \mathbf{x}^\circ(t) \tag{80}$$

Then from (78) and (79) one obtains

$$\frac{d\,\bar{\mathbf{x}}(t)}{d\,t} = \mathbf{A}\,\bar{\mathbf{x}}(t) + \mathbf{B}\,\mathbf{u}_1(t)$$

Hence, for $\mathbf{u} = \mathbf{u}_1(t)$ the system (75) becomes

$$\boldsymbol{\Sigma}_{\mathbf{A},\,\mathbf{B}} \;:\; \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \\ \mathbf{x}(\mathbf{0}) = 0 \end{cases} \qquad \mathbf{x} \in \mathbf{R}^n \tag{81}$$

Thus, the linear control system belongs to the class $\boldsymbol{\Sigma}_{\mathbf{A},\,\mathbf{B}}$ defined by (81).

5.3 Optimal Control Problems

For the given t_1 let us determine the optimal control $\tilde{\mathbf{u}}$ that extremises the functional of final values

$$\mathbf{J} = \mathbf{F}(\mathbf{x}(t_1)) = \sum_{i=1}^{n} c_i\, x_i(t_1) \tag{82}$$

and satisfies the differential constraints represented by the system (81).

The adjunct variable $\mathbf{y} \in \mathbf{R}$ satisfies equation

$$\dot{\mathbf{y}} = -\frac{\partial\,\mathbf{H}}{\partial\,\mathbf{x}} \tag{83}$$

where $\mathbf{H}$ is the Hamiltonian associated to the problem stated with respect to the optimum

$$\mathbf{H} = \langle \mathbf{y}, \dot{\mathbf{x}} \rangle \tag{84}$$

Since the final conditions $\mathbf{x}(t_1)$ are free and the final time has been indicated from the condition of transversality one obtains $\mathbf{y}(t_1) = \mathbf{y}_{t_1}$.

It results that the adjunct system becomes

$$\mathbf{\Sigma}^*_{\mathbf{A},\mathbf{B}} \; : \; \begin{cases} \dot{\mathbf{y}} = -\mathbf{A}^* \mathbf{y} \\ \mathbf{y}(t_1) = \mathbf{y}_{t_1} \end{cases} \tag{85}$$

with solution

$$\mathbf{y}(t) = \mathbf{S}^*(t_1 - t)\,\mathbf{y}_{t_1}\,, \quad \mathbf{y}_{t_1} \in \mathbf{H} \tag{86}$$

Proposition 2.

For the class of problems of optimum under consideration, in the Hilbert space there exists the identity

$$\langle \mathbf{x}(t_1), \mathbf{y}_{t_1} \rangle_{\mathbf{H}} = \int_0^{t_1} \langle \mathbf{u}, \mathbf{B}^* \mathbf{y} \rangle_{\mathbf{U}} \, d\,t \tag{87}$$

Proof

Assuming $\mathbf{u} \in \mathbf{C}^1([0, t_1], \mathbf{U})$ and $\mathbf{y}(t_1) \in \mathbf{D}(\mathbf{A}^*)$, so that $\mathbf{x}, \mathbf{y} \in \mathbf{C}^1([0, t_1], \mathbf{H})$.

Integrating by parts, since $\mathbf{x}(0) = 0$, $\mathbf{y}(t_1) = \mathbf{y}_{t_1}$, $\mathbf{B} : \mathbf{U} \to \mathbf{H}$, $\mathbf{B}^* : \mathbf{H} \to \mathbf{U}$,

$$\begin{aligned} 0 = \int_0^T \langle \dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{u}, \mathbf{y} \rangle_{\mathbf{H}} \, d\,t = \int_0^{t_1} \langle \dot{\mathbf{x}}, \mathbf{y} \rangle_{\mathbf{H}} \, d\,t - \\ \int_0^{t_1} \langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle_{\mathbf{H}} \, d\,t - \int_0^{t_1} \langle \mathbf{u}, \mathbf{B}^* \mathbf{y} \rangle_{\mathbf{U}} \, d\,t = \langle \mathbf{x}, \mathbf{y} \rangle \Big|_0^{t_1} - \\ \int_0^{t_1} \langle \mathbf{x}, -\mathbf{A}^* \mathbf{y} \rangle_{\mathbf{H}} \, d\,t - \int_0^{t_1} \langle \mathbf{x}, \mathbf{A}^* \mathbf{y} \rangle_{\mathbf{H}} \, d\,t - \int_0^{t_1} \langle \mathbf{u}, \mathbf{B}^* \mathbf{y} \rangle_{\mathbf{U}} \, d\,t \end{aligned} \tag{88}$$

One obtains

$$\langle \mathbf{x}(t_1), \mathbf{y}_{t_1} \rangle_{\mathbf{H}} - \int_0^{t_1} \langle \mathbf{u}, \mathbf{B}^* \mathbf{y} \rangle_{\mathbf{U}} \, d\,t = 0 \tag{89}$$

The identity has been demonstrated. □

This result can be extended for arbitrary $\mathbf{u} \in \mathbf{L}^2(0, t_1; \mathbf{U})$ and $\mathbf{y}_{t_1} \in \mathbf{H}$.

An important result referring to the exact controllability of the linear system (81) has been expressed by

Theorem 5.

System $\mathbf{\Sigma}_{\mathbf{A},\mathbf{B}}$ is exactly controllable $\Longleftrightarrow$ the condition is fulfilled

$$\int_0^{t_1} \| \mathbf{B}^* \mathbf{S}^*(t)\,\mathbf{y}_0 \|^2_{\mathbf{U}} \, d\,t \geq c \, \| \mathbf{y}_0 \|^2_{\mathbf{U}}\,, \quad \forall\, \mathbf{y}_0 \in \mathbf{H} \tag{90}$$

Proof

We assume that $\mathbf{\Sigma}_{\mathbf{A},\mathbf{B}}$ is exactly controllable.

Let $\mathbf{u} \in \mathbf{L}^2(0,\ t_1;\ \mathbf{U})$ and $\mathbf{y}_{t_1} \in \mathbf{H}$.

We consider that the application

$$\mathbf{L}_{t_1} : \mathbf{U} \to \mathbf{x}(T) \tag{91}$$

is self defined.

Let $\mathbf{\Lambda} : \mathbf{H} \to \mathbf{L}^2(0,\ t_1,\ \mathbf{U})$ the inverse of $\mathbf{L}_{t_1}$.

Out of Theorem 3 it results that there exists a finite dimensional subspace $\mathbf{M} \subset \mathbf{H}$, $\mathbf{M}^{\perp} = \ker \mathbf{L}_{t_1}$ so that the restriction

$$(\mathbf{L}_{t_1})_{\mathbf{M}} = (\mathbf{L}_{t_1}) \Big|_{(\ker \mathbf{L}_{t_1})^{\perp}} \tag{92}$$

is injective.

Since $\mathbf{L}_{t_1}\mathbf{u} = \mathbf{x}(t_1) \implies \mathbf{u} = \mathbf{L}_{t_1}^{-1}(\mathbf{x}(t_1)) = \mathbf{\Lambda}(\mathbf{x}(t_1))$ transfers 0 in $\mathbf{x}(t_1)$ for the system $\mathbf{\Sigma}_{\mathbf{A},\mathbf{B}}$.

We choose $\mathbf{y}(t_1) \in \mathbf{H}$ and $\mathbf{x}(t_1) = \mathbf{y}(t_1)$, $\mathbf{u} = \mathbf{\Lambda}(\mathbf{x}(t_1))$.

It results

$$\left\| \mathbf{y}_{t_1} \right\|_{\mathbf{H}}^2 = \langle \mathbf{y}_{t_1},\ \mathbf{y}_{t_1} \rangle = \int_0^{t_1} \langle \mathbf{\Lambda}(\mathbf{x}_{t_1}),\ \mathbf{B}^* \mathbf{y} \rangle_{\mathbf{U}}\ dt \tag{93}$$

For $\mathbf{\Lambda}(\mathbf{x}_{t_1}) = \mathbf{u} \in \mathbf{L}^2$, $\mathbf{B}^* \mathbf{y} \in \mathbf{L}^2$, by using Hölders's inequality one obtains

$$\begin{gathered} \left| \int_0^{t_1} (\mathbf{\Lambda}(\mathbf{x}_{t_1}))\ (\mathbf{B}^* \mathbf{y})\ dt \right| \leq \int_0^{t_1} \left| \mathbf{\Lambda}(\mathbf{x}_{t_1})\ (\mathbf{B}^* \mathbf{y}) \right|\ dt \leq \\ \left[\int_0^{t_1} \left| \mathbf{\Lambda}(\mathbf{x}_{t_1}) \right|^2\ dt \right]^{1/2} \cdot \left[\int_0^{t_1} |\mathbf{B}^* \mathbf{y}|^2\ dt \right]^{1/2} \end{gathered} \tag{94}$$

Out of relations (93) and (94) for $\mathbf{x}_{t_1} = \mathbf{y}_{t_1}$, we have

$$\begin{gathered} \left\| \mathbf{y}_{t_1} \right\|_{\mathbf{H}}^2 \leq \left\| \mathbf{\Lambda}\ (\mathbf{y}_{t_1}) \right\| \left(\int_0^{t_1} \| \mathbf{B}^* \mathbf{y} \|_{\mathbf{U}}^2\ dt \right)^{1/2} \leq \\ \| \mathbf{\Lambda} \|\ \left\| \mathbf{y}_{t_1} \right\|_{\mathbf{H}} \left(\int_0^{t_1} \| \mathbf{B}^* \mathbf{y} \|_{\mathbf{U}}^2\ dt \right)^{1/2} \end{gathered} \tag{95}$$

or

$$\int_0^{t_1} \| \mathbf{B}^* \mathbf{y} \|_{\mathbf{U}}^2\ dt \geq \frac{1}{\| \mathbf{\Lambda} \|^2} \left\| \mathbf{y}_{t_1} \right\|_{\mathbf{H}}^2 \tag{96}$$

By changing $t \to t_1 - t \implies \mathbf{y}_0$ becomes $\mathbf{y}_{t_1}$ and $\mathbf{S}^*(t)$ is transformed into $\mathbf{S}^*(t_1 - t)$.

It results one equivalent relation of controllable system in which $\mathbf{y}_{t_1}$ substitute $\mathbf{y}_0$.

$$\int_0^{t_1} \| \mathbf{B}^* \mathbf{S}^*(t)\, \mathbf{y}_0 \|^2_{\mathbf{U}} \, dt = \int_0^{t_1} \| \mathbf{B}^* \mathbf{S}^*(t_1 - t)\, \mathbf{y}_0 \|^2_{\mathbf{U}} \, dt \geq \frac{1}{\| \mathbf{\Lambda} \|^2} \left\| \mathbf{y}_{t_1} \right\|^2_{\mathbf{H}} = c \left\| \mathbf{y}_{t_1} \right\|^2_{\mathbf{H}} \tag{97}$$

The direct implication has been demonstrated.

$\Longleftarrow$ We assume that the condition has been fulfilled.

$$\int_0^{t_1} \| \mathbf{B}^* \mathbf{y} \|^2_{\mathbf{U}} \, dt \geq c \left\| \mathbf{y}_{t_1} \right\|^2_{\mathbf{H}} \tag{98}$$

For any $\mathbf{y}_{t_1} \in \mathbf{H}$ we take the multitude $\{ \mathbf{u}(t) = \mathbf{B}^* \mathbf{y}(t) \}$ ($\mathbf{y}(t)$ is solution of $\mathbf{\Sigma}^*_{\mathbf{A},\mathbf{B}}$).

We consider the solution $\mathbf{x}(t)$ for $\mathbf{\Sigma}_{\mathbf{A},\mathbf{B}}$ which corresponds to the control $\mathbf{u}(t)$ mentioned.

One defines the bounded operator

$$\mathbf{\Gamma} \,:\, \mathbf{y}_{t_1} \in \mathbf{H} \,\rightarrow\, \mathbf{x}(t_1) = \mathbf{L}_{t_1} \left(\mathbf{B}^* \mathbf{y}(\,\cdot\,) \right) \in \mathbf{H} \tag{99}$$

Since $\mathbf{u}(t) = \mathbf{B}^* \mathbf{y}$, the identity (89) becomes

$$\left\langle \mathbf{\Gamma} \mathbf{y}_{t_1}, \mathbf{y}_{t_1} \right\rangle_{\mathbf{H}} = \int_0^{t_1} \| \mathbf{B}^* \mathbf{y}(t) \|^2_{\mathbf{U}} \, dt \geq c \left\| \mathbf{y}_{t_1} \right\|^2_{\mathbf{H}} \tag{100}$$

Therefore, there is a constant $c > 0$ for which (100) is satisfied. It follows that $\left\langle \mathbf{\Gamma} \mathbf{y}_{t_1}, \mathbf{y}_{t_1} \right\rangle_{\mathbf{H}}$ is positively defined.

This resumes the conclusion that the system $\mathbf{\Sigma}_{\mathbf{A},\mathbf{B}}$ is controllable.

Thus, since $\mathbf{\Gamma}$ is irreversible, for any $\mathbf{x}_{t_1} \in \mathbf{H}$, state $\mathbf{y}_{t_1} = \mathbf{\Gamma}^{-1}(\mathbf{x}_{t_1})$ is such that there exists control $\mathbf{u}(t) = \mathbf{B}^* \mathbf{y}(t)$ which transfers 0 in $\mathbf{x}_{t_1}$.

The theorem has been demonstrated. □

6. Conclusions

The present paper analyzes the existence and uniqueness of optimal control in issues regarding the minimization of time and energy for controllable linear systems. At the same time, one has demonstrated the necessary and sufficient condition of exact controllability for Bolza functionals in problems of optimal transfer.

The mathematical model proposed uses the results of the functional analysis in Banach and Hilbert spaces, together with the properties of bounded linear operators.

In section 5.2 the nonhomogeneous linear control systems are transformed into homogeneous linear systems with a null initial condition.

For the control of this class of systems it is necessary to use the solution of the adjoint system corresponding to the problem of optimal transfer stated (Sections 5.1 and 5.3).

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