# Minimum Energy for Controllable Nonlinear Systems

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**Abstract.** Let us consider the controllable nonlinear system $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u} + \mathbf{F}(\mathbf{y}), \mathbf{y}(0) = 0$. One uses the results regarding the controllability operator $\mathbf{Q}_t$ for the linear system. Thus, one defines a topology that ensures the evaluation of the admissible domain of minimum energy. One demonstrates the convergence of the solution for the class of nonlinear systems under analysis. Energy minimization involves the determination of a single minimizing element in the topology under consideration.

**Keywords**: controllable nonlinear systems, minimum energy, topology.

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## 1. Introduction

The study of linear systems $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ whose controllability is null and accurate highlighted some features related to minimum energy. [1], [4], [5], [11], [12]. The controllability oerator $\mathbf{Q}$ is the unique solution of a differential equation. [1], [5], [6], [8], [14]. If $\mathbf{A}$ engenders a stable semi group there exists $\lim_{t\to\infty} \mathbf{Q}_t = \mathbf{Q}$ which is the only solution of a Liapunov equation. [1], [5], [14].

For null controllable linear systems one determines the energy at infinity. [1], [5], [14]. One also assesses the domain of minimum energy based on the hypothesis of the stability of the uncontrolled system. In [2], [3], [11], [15] one demonstrates the uniqueness of the issues involving a minimum norm in the case of nonlinear systems. One obtains the conditions of the existence of the solution of the iterative process regarding the integration of the equivalent system expressed through the linear operator defined by the solution of the homogeneous system.

## 2. Linear control systems

We consider the control linear system

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u}, \quad \mathbf{y}(0) = \mathbf{a} \in \mathbf{H} \tag{1}$$

on the Hilbert space H.

Where $\mathbf{A}$, $\mathbf{B}$ are constants matrices of dimensions $n \times n$, $\mathbf{B}$ is nonsingular and $\mathbf{u} \in \mathbf{U}$ (Hilbert space) is $n \times 1$ vector.

Operator $\mathbf{A}$ generates the semigroup $\mathbf{C}_0$ of the linear operator $\mathbf{S}(t), t \geq 0$ while $\mathbf{B}$ is a Hilbert space $\mathbf{H}$ operator at $\mathbf{H}$. We assume that $\mathbf{u} \in \mathbf{L}^2[0, T; \mathbf{U}]$ for an arbitrary $T > 0$. Let us consider for $T > 0$ operator $\mathbf{L}_T : \mathbf{L}^2[0, T; \mathbf{U}] \to \mathbf{H}$

$$\mathbf{L}_T\mathbf{u} = \int_0^T \mathbf{S}(T - s)\,\mathbf{B}\,\mathbf{u}(s)\,d\,s \tag{2}$$

The solution for system (1) is given by

$$\mathbf{y}(t) = \mathbf{S}(t)\,\mathbf{a} + \int_0^t \mathbf{S}(t-s)\,\mathbf{B}\,\mathbf{u}(s)\,d\,s \tag{3}$$

or

$$\mathbf{y}(t) = \mathbf{S}(t)\,\mathbf{a} + \mathbf{L}_T\mathbf{u} \tag{4}$$

There exists $\mathbf{u}(\cdot) \in \mathbf{L}^2[0, T; \mathbf{U}]$ that transfers $\mathbf{a}$ into $\mathbf{b}$ at time $T$, if and only if $\mathbf{b} - \mathbf{S}(T)\,\mathbf{a} \in Im\,\mathbf{L}_T$. System (1) is null controllable at time $T > 0$ if an arbitrary state $\mathbf{b} \in \mathbf{H}$ can be transferred into 0 at time $T$.

We define the linear operator

$$\mathbf{Q}_t = \int_0^t \mathbf{S}(r)\,\mathbf{B}\,\mathbf{B}^*\,\mathbf{S}^*(r)\,d\,r \tag{5}$$

The results obtained are summed up via:

**Proposition 1.**

1. Function $\mathbf{Q}_t$, $t \geq 0$ is the unique solution of the equation

$$\begin{gathered}\frac{d}{dt} < \mathbf{Q}_t\,\mathbf{x},\ \mathbf{x} >= 2\ < \mathbf{Q}_t\,\mathbf{A}\,\mathbf{x},\ \mathbf{x} > + \|\,\mathbf{B}^*\mathbf{x}\,\|^2, \\ \mathbf{x} \in \mathbf{D}(\mathbf{A}^*),\ t \geq 0,\ \mathbf{Q}_0 = \mathbf{I}\end{gathered} \tag{6}$$

2. If $\mathbf{A}$ generates a stable semigroup, then

$$\lim_{t\to\infty} \mathbf{Q}_t = \mathbf{Q} \tag{7}$$

exists and it is the only solution of the equation

$$2\ < \mathbf{Q}\,\mathbf{A}\,\mathbf{x},\ \mathbf{x} > + \|\,\mathbf{B}^*\mathbf{x}\,\|^2 = 0, \quad \mathbf{x} \in \mathbf{D}(\mathbf{A}^*) \tag{8}$$

where

$$\mathbf{D}(\mathbf{A}^*) = \{\mathbf{x} \in \mathbf{H}\ |\ \exists\,\mathbf{C} \in \mathbf{R}^+,\ |\mathbf{A}\,\mathbf{y},\ \mathbf{x}\,| \leq \mathbf{C}\,\|\,\mathbf{y}\,\|_{\mathbf{H}},\ \forall\,\mathbf{y} \in \mathbf{D}(\mathbf{A})\} \tag{9}$$

and we define

$$< \mathbf{A}\,\mathbf{y},\ \mathbf{x} > = < \mathbf{y},\ \mathbf{A}^*\mathbf{x} >_{\mathbf{H}}, \forall\,\mathbf{y} \in \mathbf{D}(\mathbf{A}), \forall\,\mathbf{x} \in \mathbf{D}(\mathbf{A}^*) \tag{10}$$

**Theorem 1**.

1. For $T > 0$ arbitrary and $\mathbf{a}, \mathbf{b} \in \mathbf{H}$

$$\mathbf{E}_T(\mathbf{a},\ \mathbf{b}) = \left\| \left(\mathbf{Q}_T^{1/2}\right)^{-1}\ (\mathbf{S}(T)\,\mathbf{a} - \mathbf{b}) \right\|^2 \tag{11}$$

2. If $\mathbf{S}(t)$ is stable and the system (1) is null controllable during time $T_0 > 0$, then

$$\mathbf{E}_\infty(0,\ \mathbf{b}) = \left\| \left(\mathbf{Q}_T^{1/2}\right)^{-1}\ \mathbf{b} \right\|^2 \tag{12}$$

3. There exist $\mathbf{C}_T > 0$, so that

$$\left\| \left(\mathbf{Q}_T^{1/2}\right)^{-1}\ \mathbf{b} \right\|^2 \leq \mathbf{E}_T(0,\ \mathbf{b}) \leq \mathbf{C}_T\ \left\| \left(\mathbf{Q}_T^{1/2}\right)^{-1}\ \mathbf{b} \right\|^2 \tag{13}$$

## 3. Nonlinear controllable systems

Let space

$$\mathbf{V}_T = Im\mathbf{L}_T = Im\mathbf{Q}_T^{1/2} \tag{14}$$

For $\mathbf{x} \in \mathbf{V}_T$ one has

$$\mathbf{x} = \mathbf{L}_T\, \mathbf{x}_T \tag{15}$$

Thus space V associated with the system control (1) is a normed space where

$$\|\mathbf{x}\|_T = \|\mathbf{L}_T^{-1}\mathbf{x}\| = \left\|\left(\mathbf{Q}_T^{1/2}\right)^{-1}\mathbf{x}\right\| \tag{16}$$

If $t \le s$, as $\left(\mathbf{Q}_t^{1/2}\right)^{-1}$ is a decreasing function, out of (14) and (16) there results

$$\mathbf{V}_t \in \mathbf{V}_S\,, \quad \|\mathbf{x}\|_S = \left\|\left(\mathbf{Q}_T^{1/2}\right)^{-1}\mathbf{x}\right\| \le \left\|\left(\mathbf{Q}_t^{1/2}\right)^{-1}\mathbf{x}\right\| = \|\mathbf{x}\|_t, \mathbf{x} \in \mathbf{V}_t \tag{17}$$

We consider the nonlinear system

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{F}(y) + \mathbf{B}\mathbf{u}, \quad \mathbf{y}(0) = 0 \tag{18}$$

or

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{B}^{-1}\mathbf{F}(y) + \mathbf{B}\mathbf{u}, \quad \mathbf{y}(0) = 0 \tag{19}$$

By writing

$$\mathbf{B}^{-1}\mathbf{F}(y) = \mathbf{G}(y) \tag{20}$$

the system (19) becomes

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{G}(y) + \mathbf{B}\mathbf{u}, \quad \mathbf{y}(0) = 0 \tag{21}$$

If $\mathbf{x} \in \mathbf{V}_T$, $\mathbf{V}_T = Im\mathbf{L}_T = \mathbf{L}_T\,\mathbf{x}$ and $\mathbf{L}_T : \mathbf{L}^2[0, T; \mathbf{U}]$ one gets

$$\mathbf{L}_T^{-1} : \mathbf{V}_T \to \mathbf{U} \tag{22}$$

We assume that for every $T > 0$ sufficiently small, $\mathbf{G} : \mathbf{V}_T \to \mathbf{U}$ and for any $r > 0$, there exists $\mathbf{M}_{r,T}$, so that

$$\|\mathbf{G}(a) - \mathbf{G}(b)\|_{\mathbf{U}} \le \mathbf{M}_{r,T}\,\|\mathbf{a} - \mathbf{b}\|_T \tag{23}$$

featuring

$$\|\mathbf{a}\|_T \le r\,, \qquad \|\mathbf{b}\|_T \le r \tag{24}$$

The nonlinear system (21) allows the solution

$$\mathbf{y}(t) = \int_0^t \mathbf{S}(t-r)\,\mathbf{B}\,\mathbf{G}(r)\,d\,r + \int_0^t \mathbf{S}(t-r)\,\mathbf{B}\,\mathbf{u}(r)\,d\,r \tag{25}$$

In further developments we use

**Proposition 2.**

Solution $\mathbf{y}(\cdot)$ with the initial zero condition yielded by (25) is $\mathbf{V}_T$ continuous on $[0,\ T]$

*Proof*

For $0 \leq t \leq s \leq T$ established, one has

$$\mathbf{y}(s) - \mathbf{y}(t) = \int_0^s \mathbf{S}(s-r)\mathbf{B}\,\mathbf{u}(r)\,d\,r - \int_0^t \mathbf{S}(t-\tau)\mathbf{B}\,\mathbf{u}(\tau)d\,\tau \tag{26}$$

Writing the second integral within the same boundaries as the first integral is done by choosing $t - \tau = s - r$. One gets

$$\begin{aligned} &\tau = r - (s-t) \\ &\tau_1 = 0 \rightarrow r_1 = s - t \\ &\tau_2 = 0 \rightarrow r_2 = 0 \end{aligned} \tag{27}$$

Thus, by implementing this change, relation (26) becomes

$$\begin{aligned} \mathbf{y}(s) - \mathbf{y}(t) = \int_0^s \mathbf{S}(s-r)\,\mathbf{B}\,\mathbf{u}(r)\,d\,r - \int_{s-t}^s \mathbf{S}(s-r)\,\mathbf{B}\,\mathbf{u}(r-(s-t))\,d\,r = \\ \int_0^s \mathbf{S}(t-r)\,\mathbf{B}\,\left[\,\mathbf{u}(r) - \mathbf{u}(r-(s-t))\,\mathbf{I}_{[s-t,\,s]}(r)\,\right]\,d\,r \end{aligned} \tag{28}$$

where

$$\mathbf{I}_{[s-t,\,s]} = \begin{cases} 0 & r \in [0,\ s-t) \\ 1 & r \in (s-t,\ s] \end{cases} \tag{29}$$

We define

$$\mathbf{L}_S\,\mathbf{u} = \int_0^s \mathbf{S}(s-r)\mathbf{B}\,\mathbf{u}(r)\,d\,r \tag{30}$$

By taking into consideration the definition of the norm in $\mathbf{V}_S$ and $\mathbf{V}_T$ and by using (17), (25) results

$$\begin{aligned} \|\,\mathbf{y}(s) - \mathbf{y}(t)\,\|_T^2 \leq \|\,\mathbf{y}(s) - \mathbf{y}(t)\,\|_S^2 = \\ \left\| \int_0^s \mathbf{S}(s-r)\,\mathbf{B}\,\left[\,\mathbf{u}(r) - \mathbf{u}(r-(s-t))\,\mathbf{I}_{[s-t,\,s]}(r)\,\right]\,d\,r \right\|^2 \end{aligned} \tag{31}$$

For $p \in (1, \infty)$, $q = p/(p-1)$ we consider spaces $\mathbf{L}^p$ and $\mathbf{L}^q$ where

$$\begin{aligned} &\mathbf{S}(s-r)\,\mathbf{B} \in \mathbf{L}^2, \\ &\{\,\mathbf{u}(r) - \mathbf{u}(r-(s-t))\,\mathbf{I}_{[s-t,\,s]}(r)\,\} \in \mathbf{L}^2\,,\ \ (p = 2,\ q = 2) \end{aligned} \tag{32}$$

By applying Holder's inequality we have

$$\left\| \int_0^s \mathbf{S}(s-r)\,\mathbf{B}\,[\,\mathbf{u}(r) - \mathbf{u}(r-(s-t))\,\mathbf{I}_{[s-t,\,s]}(r)\,]\;d\,r \right\| \leq$$

$$\left[ \int_0^s \|\mathbf{S}(s-r)\,\mathbf{B}\,\|^2\,d\,r \right]^{1/2} \cdot \tag{33}$$

$$\left[ \int_0^s \left\| \mathbf{u}(r) - \mathbf{u}(r-(s-t))\,\mathbf{I}_{[s-t,\,s]}(r) \right\|\;d\,r \right]^{1/2}$$

Thus (31) becomes

$$\|\,\mathbf{y}(s) - \mathbf{y}(t)\,\|_T^2 \leq \left( \int_0^s \|\mathbf{S}(s-r)\,\mathbf{B}\,\|^2\,d\,r \right) \cdot$$

$$\left( \int_0^s \left\| \mathbf{u}(r) - \mathbf{u}(r-(s-t))\,\mathbf{I}_{[s-t,\,s]}(r) \right\|\;d\,r \right) \tag{34}$$

The evaluation of the right member in (34) is done by using the consequences of Hahn-Banach Theorem.

**Theorem 2.**

Let **X** be a normed linear space and let $\mathbf{x} \in \mathbf{X}$, $\mathbf{x} \neq 0 \implies$

There exist $f \in \mathbf{X}^n$ (the dual of **X**) so that

$$\mathbf{f}(x) = \|\,\mathbf{x}\,\| \tag{35}$$

and

$$\|\,\mathbf{f}\,\| = 1 \tag{36}$$

*Proof*

We consider that **X** is the space with a scalar product

$$\mathbf{f}(x) = < x,\, y > \tag{37}$$

where

$$\mathbf{y} = \frac{\mathbf{x}_1}{\|\,\mathbf{x}_1\,\|} \tag{38}$$

Thus, we can write

$$\mathbf{f}(x_1) = \left\langle x_1,\, \frac{\mathbf{x}_1}{\|\,\mathbf{x}_1\,\|} \right\rangle = \frac{1}{\|\,\mathbf{x}_1\,\|}\; < x_1,\, x_1 > = \|\,\mathbf{x}_1\,\| \tag{39}$$

There results

$$\|\,\mathbf{f}\,\| = \sup \frac{\mathbf{f}(x_1)}{\|\,\mathbf{x}_1\,\|} = 1 \tag{40}$$

The theorem has been demonstrated. □

Let

$$\mathbf{x} \in \mathbf{X} = \{ \mathbf{S}(s-r)\,\mathbf{B},\ r \in [0,\ s] \} \tag{41}$$

By applying Theorem 2 to the space $\mathbf{L}^2[0,\ s]$ where

$$\|\mathbf{f}\| = \left( \int_0^s |\mathbf{f}(t)|^2\, d\,t \right)^{1/2} \tag{42}$$

and by taking

$$\mathbf{f}(x_1) = \|\mathbf{S}(s-r)\,\mathbf{B}\| = |\mathbf{x}_1\| \tag{43}$$

we get

$$\|\mathbf{f}\|^2 = \int_0^s \|\mathbf{S}(s-r)\,\mathbf{B}\|^2\ d\,r = 1 \tag{44}$$

From (31) a(34), (44) one obtains

$$\begin{gathered} \|\mathbf{y}(s)-\mathbf{y}(t)\|_T^2 \le \|\mathbf{y}(s)-\mathbf{y}(t)\|_S \le \int_0^s \|\mathbf{u}(r)-\mathbf{u}(r-(s-t)))\|^2\ d\,r = \\ \int_0^{s-t} \|\mathbf{u}(r)\|^2\ d\,r + \int_{s-t}^{t} \|\mathbf{u}(r)-\mathbf{u}(r-(s-t)))\|^2\ d\,r \end{gathered} \tag{45}$$

It results that for $(s-t) \to \infty$, $\|\mathbf{y}(s)-\mathbf{y}(t)\|_T^2 \to \infty$. Hence $\mathbf{y}(t)$ of the nonlinear system under consideration is continuous $\mathbf{V}_T$ on $[0,\ T]$.

Proposition 2 has been demonstrated. □

This result is useful in evaluating minimum energy $\mathbf{E}_T^N$.

**Theorem 3.**

If there are fulfilled the conditions

$$\begin{gathered} 2\,\mathbf{M}_{r,T}\ \sqrt{T} < 1 \quad \text{and} \\ \|\mathbf{b}\|_T \le r\left(1 - 2\,\mathbf{M}_{r,T}\ \sqrt{T}\right) \end{gathered} \tag{46}$$

$\Longrightarrow$ Minimum energy of the nonlinear system (41) $\mathbf{E}_T^N$ is unique and satisfies

$$\left(\|\mathbf{b}\|_T - r\,\mathbf{M}_{r,T}\ \sqrt{T}\right)^2 \le \mathbf{E}_T^N \le \left(\|\mathbf{b}\|_T + r\,\mathbf{M}_{r,T}\ \sqrt{T}\right)^2 \tag{47}$$

Solution (25) of nonlinear system (21) can be written

$$\mathbf{y}(t) = \mathbf{L}_t\,\mathbf{G}(y) + \mathbf{L}_t\,\mathbf{u} \tag{48}$$

where

$$\mathbf{L}_t = \int_0^t \mathbf{S}(t-r)\,\mathbf{B}\,d\,r \tag{49}$$

and

$$\mathbf{G}(y) = \mathbf{G}(y(s)),\quad s \in [0,\ T] \tag{50}$$

Let $\mathbf{x} \in \mathbf{V}_t \subset \mathbf{V}_T$.

If there is a control $\mathbf{u}(\cdot)$ which transfers zero in $\mathbf{b}$ at time $T$, then

$$\mathbf{x} = \mathbf{L}_T\, \mathbf{G}(y) + \mathbf{L}_T\, \mathbf{u} \tag{51}$$

where $\mathbf{u}(\cdot)$ one obtains the condition $\mathbf{y}(T) = \mathbf{b}$, it result

$$\mathbf{u} = \mathbf{L}_T^{-1}\,(\mathbf{b} - \mathbf{L}_T\, \mathbf{G}(y)) \tag{52}$$

With control (52) which transfers zero in $\mathbf{b}$, solution of nonlinear system (48) on $\mathbf{V}_T$, becomes

$$\mathbf{y}(t) = \mathbf{L}_t\, \mathbf{G}(y) + \mathbf{L}_t\, \mathbf{L}_T^{-1}\,(\mathbf{b} - \mathbf{L}_T\, \mathbf{G}(y)),\ t \in [0,\ T] \tag{53}$$

From Proposition 2. this relation is continuous $\mathbf{V}_T$ for control (52). Thus we get

$$\lim_{t \to T} \mathbf{y}(t) = \mathbf{y}(T) \tag{54}$$

we have

$$\mathbf{y}(T) = \mathbf{L}_t\, \mathbf{G}(y) + \mathbf{x} - \mathbf{L}_t\, \mathbf{G}(y) = \mathbf{x} \tag{55}$$

*Proof of Theorem 3*

For $\omega \in \Omega = \mathbf{C}\,[0,\ T;\ \mathbf{V}_T]$ the space of continuous functions on $[0,\ T]$ from $\mathbf{V}_T$, we consider $\Phi\ :\ \Omega \longrightarrow \Omega$, defined by

$$\Phi(\omega)(t) = \mathbf{L}_t\, \mathbf{G}(\omega) + \mathbf{L}_t\, \mathbf{L}_T^{-1}\,(\mathbf{x} - \mathbf{L}_T\, \mathbf{G}(\omega)) \tag{56}$$

With $\omega^\circ = 0$ which represents the initial condition, we write

$$\Phi(0)(t) = \mathbf{L}_t\, \mathbf{L}_T^{-1}\, \mathbf{x}, \quad t \in [0,\ T] \tag{57}$$

As we have

$$\begin{gathered} \sup_{t \le T} \left\| \mathbf{L}_t\, \mathbf{L}_T^{-1}\, \mathbf{b} \right\|_T^2 = \left\| \int_0^T \mathbf{S}(T - r)\, \mathbf{B} \left( \mathbf{L}_T^{-1}\, \mathbf{b} \right) d\,r \right\|_U^2 \le \\ \int_0^T \| \mathbf{S}(T - r)\, \mathbf{B} \|_U^2 \cdot \int_0^T \left\| \mathbf{L}_T^{-1}\, \mathbf{b}(s) \right\|_U^2 \end{gathered} \tag{58}$$

Out of (44) and by using the topology adopted

$$\left\| \mathbf{L}_T^{-1}\, \mathbf{x} \right\| = \| \mathbf{x} \|_T = \| \mathbf{b} \|_T = \left\| \mathbf{L}_T^{-1}\, \mathbf{b} \right\| \tag{59}$$

the relation (58) becomes

$$\sup_{t \le T} \left\| \mathbf{L}_t\, \mathbf{L}_T^{-1}\, \mathbf{b} \right\|_T^2 \le \int_0^T \left\| \mathbf{L}_T^{-1}\, \mathbf{b}(s) \right\|_U^2\, d\,s = \| \mathbf{b} \|_T^2 \tag{60}$$

which is written

$$\| \Phi(0) \|_\Omega \le \| \mathbf{b} \|_T \tag{61}$$

Let $\bar{\omega},\ \omega \in \Omega$, so that from (56) one obtains

$$\begin{gathered}\|\,\Phi(\bar{\omega})(t)-\Phi(\omega)(t)\,\|_{\Omega} \le \|\,\mathbf{L}_t\,(G(\bar{\omega})-G(\omega))\,\|_{\Omega} + \\ \left\|\,\mathbf{L}_t\,\mathbf{L}_T^{-1}\,\left(\mathbf{L}_t\,\left[(G(\bar{\omega})-G(\omega)]\,\right)\right\|_{\Omega} = 2\,\|\,\mathbf{L}_t\,(G(\bar{\omega})-G(\omega))\,\|_{\Omega} \le \\ 2\left(\int_0^T \|\,G(\bar{\omega}(s))-G(\omega(s))\,\|_U^2\,d\,s\right)^{1/2}\end{gathered} \tag{62}$$

It follows that if $\|\,\bar{\omega}\,\|_{\Omega} \le r$, $\|\,\omega\,\|_{\Omega} \le r$, then

$$\begin{gathered}\|\,\Phi(\bar{\omega})(t)-\Phi(\omega)(t)\,\|_{\Omega} \le \\ 2\,\mathbf{M}_{r,\,T}\left[\int_0^T \|\,\bar{\omega}(s)-\omega(s)\,\|_T^2\,d\,s\right]^{1/2} \le 2\,\mathbf{M}_{r,\,T}\,\sqrt{T}\,\|\,\bar{\omega}-\omega\,\|_{\Omega}\end{gathered} \tag{63}$$

or by noting

$$\|\,\Phi(\bar{\omega})(t)-\Phi(\omega)(t)\,\|_{\Omega} \le k\,\|\,\bar{\omega}-\omega\,\|_{\Omega} \tag{64}$$

Solving the nonlinear system (56) involves an iterative process for which the successive approximations determine a convergent series.

Let $\omega_i(t) = \Phi^{(i)}(t,\ \omega_{i-1})\ \ i = 1,\ 2,\ \ldots$ the successive approximations of the iterative process with the initial data $\omega^0 = \omega(0) = 0\ (\omega_j^0 = 0)\ \ j = 1,\ \ldots,\ n$ and $\omega_i(t) = \left(\omega_i^1,\ \ldots,\ \omega_i^1\right)$.

The initial conditions represent approximations of the zero order.

We write the successive approximations for all the unknown functions.

The approximation of the first order is given by

$$\omega_1(t) = \Phi^{(1)}(t,\ \omega^0) = \Phi(0) \tag{65}$$

The approximation of the second order is obtained from the approximation of the first order

$$\omega_2(t) = \Phi^{(2)}(t,\ \omega_1) \tag{66}$$

Generally, the approximation of order $m$ is determined by using the approximation of order $(m-1)$, so that

$$\omega_m(t) = \Phi^{(m)}(t,\ \omega_{m-1}) \tag{67}$$

We shall demonstrate that the successive approximations form a convergent series, hence there exists $\lim\limits_{m\to\infty}\ \omega_m$.

Thus, to consider series

$$\begin{gathered}\omega^0 + \left[\,\omega_1(t)-\omega^0\,\right] + [\,\omega_2(t)-\omega_1(t)\,] + \ \ldots,\ + \\ [\,\omega_m(t)-\omega_{m-1}(t)\,] + \ \ldots\end{gathered} \tag{68}$$

The evaluation of the series terms is carried out inductively, by using the Lipshitz condition (64).

One obtains

$$\begin{gathered}\|\,\omega_1(t)\,\|_{\Omega} = \left\|\,\Phi^{(1)}(t,\ \omega^0)\,\right\|_{\Omega} = \|\,\Phi(0)\,\|_{\Omega} \\ \|\,\omega_2(t)-\omega_1(t)\,\|_{\Omega} = \left\|\,\Phi^{(2)}(t,\ \omega_1)-\Phi^{(1)}(t,\ \omega^0)\,\right\|_{\Omega} \le \\ k\,\|\,\omega_1-0\,\|_{\Omega} = k\,\|\,\Phi(0)\,\|_{\Omega} \\ \|\,\omega_3(t)-\omega_2(t)\,\|_{\Omega} = \left\|\,\Phi^{(3)}(t,\ \omega_2)-\Phi^{(2)}(t,\ \omega_1)\,\right\|_{\Omega} \le \\ k\,\cdot\,k\,\|\,\omega_2-\omega_1\,\|_{\Omega} = k^2\,\|\,\Phi(0)\,\|_{\Omega}\end{gathered} \tag{69}$$

We assume that for the term $(\omega_{m-1}(t) - \omega_{m-2}(t))$ we have

$$\begin{gathered} \| \omega_{m-1}(t) - \omega_{m-2}(t) \|_\Omega = \left\| \Phi^{(m-1)}(t,\, \omega_{m-2}) - \Phi^{(m-2)}(t,\, \omega_{m-3}) \right\|_\Omega \leq \\ k^{m-2} \| \omega_{m-2} - \omega_{m-3} \|_\Omega = k^{m-2} \, \| \Phi(0) \|_\Omega \end{gathered} \tag{70}$$

We are going to show that a similar evaluation where $(m - 1)$ is replaced by m is valid for the following term.

Indeed, we have

$$\begin{gathered} \| \omega_m(t) - \omega_{m-1}(t) \|_\Omega = \left\| \Phi^{(m)}(t,\, \omega_{m-1}) - \Phi^{(m-1)}(t,\, \omega_{m-2}) \right\|_\Omega \leq \\ k \, \| \omega_{m-1} - \omega_{m-2} \|_\Omega = k \cdot k^{m-2} \, \| \Phi(0) \|_\Omega = k^{m-1} \, \| \Phi(0) \|_\Omega \end{gathered} \tag{71}$$

It follows that the evaluation (71) is valid for any natural number $m$.

One notices that all the series terms (68) starting form the second are lower/smaller or equal in absolute value to the terms of the geometrical series $\sum_{m=1}^{\infty} k^{m-1} \| \Phi(0) \|$, convergent when ration $k = 2\,\mathbf{M}_{r,T} \sqrt{T} < 1$.

The series (68) is convergent in $\Omega$. The terms of these series being continuous and partial sums are continuous.

The series $\omega_n = \Phi^{(n)}(t,\, 0)$ is convergent at $\Omega$ in the solution of equation (53).

Thus for $\Omega$ we can write

$$\mathbf{y}(t) = \sum_{s=1}^{\infty} \left[ \Phi^{(s)}(t,\, 0) - \Phi^{(s-1)}(t,\, 0) \right] = \lim_{n \to \infty} \Phi^{(n)}(t,\, 0) \tag{72}$$

The solution $\mathbf{y}(t)$ satisfies the initial condition $\mathbf{y}(0) = 0$ on $\Omega$.

Indeed, since

$$\omega_m(t) = \Phi^{(m)}(t,\, \omega_{m-1}(t)) \qquad m \in \mathbf{N} \tag{73}$$

or

$$\omega_m(0) = \Phi^{(m)}(0,\, \omega_{m-1}(0)) = \Phi^{(m)}(0,\, 0) \qquad m \in \mathbf{N} \tag{74}$$

wherefrom

$$\lim_{n \to \infty} \Phi^{(m)}(0,\, 0) = \mathbf{y}(0) = 0 \tag{75}$$

For an approximation of $n$ order, from (68) and (69) one obtains

$$\begin{gathered} \left\| \Phi^{(n)}(t,\, 0) \right\|_\Omega \leq \| \Phi(0) \| + \sum_{s=1}^{\infty} \left\| \Phi^{(s)}(t,\, 0) - \Phi^{(s-1)}(t,\, 0) \right\|_\Omega \\ \leq \left( 1 + K + k^2 + \ldots k^{n-1} \right) \| \Phi(0) \|_\Omega \end{gathered} \tag{76}$$

There results

$$\left\| \Phi^{(n)}(t,\, 0) \right\|_\Omega \leq \frac{1}{1-k} \, \| \Phi(0) \|_\Omega \leq \frac{1}{1-k} \, \| \mathbf{b} \|_T \tag{77}$$

If there is fulfilled the condition

$$\frac{1}{1 - 2\,\mathbf{M}_{r,T} \sqrt{T}} \, \| \mathbf{b} \|_T \leq r \tag{78}$$

then $\left\| \Phi^{(n)}(t,\, 0) \right\|_{\Omega} \leq r$ and the series $\{ \omega_n(t) \}$ is bounded hence convergent and there exists $\lim\limits_{n \to \infty} \omega_n$ .

The energy of the nonlinear system in the chosen topology becomes

$$\| \mathbf{u} \|_{\mathbf{L}^2(0,\, T;\, U)} = \left[ \int_0^T \| \mathbf{u} \|^2 \; d\, s \right]^{1/2} = \left\| \mathbf{L}_T^{-1} (\mathbf{b} - \mathbf{L}_T\, \mathbf{G}(y)) \right\|_T = \| \mathbf{b} - \mathbf{L}_T\, \mathbf{G}(y) \|_T \tag{79}$$

There results the existence of the unique minimum energy by using the theorem of the unique minimizing element:

**Theorem 4.**

Let $\mathbf{H}$ be the Hilbert space and $\mathbf{H}_1$ a closed subspace of $\mathbf{H}$ and $\mathbf{b} \in \mathbf{H}$, $\mathbf{b} \notin \mathbf{H}_1$. $\Longrightarrow$ There is an element $\mathbf{z} \in \mathbf{H}_1$ and a real number $d \in \mathbf{R}_+$, so that

$$\| \mathbf{b} - \mathbf{z} \| = \inf_{w \in \mathbf{H}_1} \| \mathbf{b} - \mathbf{w} \| \tag{80}$$

*Proof*

We consider a sequence $\{ \mathbf{w}_i \} \in \mathbf{H}_1$ for which

$$\| \mathbf{b} - \mathbf{w}_i \| \longrightarrow d \tag{81}$$

Let

$$\mathbf{w}_m,\;\; \mathbf{w}_n \in \{ \mathbf{w}_i \} \tag{82}$$

By applying the parallelogram law to the Hilbert space we get

$$\begin{gathered} \|(\mathbf{b} - \mathbf{w}_n) + (\mathbf{b} - \mathbf{w}_m)\|^2 + \| (\mathbf{b} - \mathbf{w}_n) + (\mathbf{b} - \mathbf{w}_m) \|^2 = \\ 2\, \| \mathbf{b} - \mathbf{w}_n \|^2 + 2\, \| \mathbf{b} - \mathbf{w}_m \|^2 \end{gathered} \tag{83}$$

This yields

$$4 \left\| \mathbf{b} - \frac{\mathbf{w}_n + \mathbf{w}_m}{2} \right\|^2 + \| \mathbf{w}_m - \mathbf{w}_n \|^2 = 2\, \| \mathbf{b} - \mathbf{w}_n \|^2 + 2\, \| \mathbf{b} - \mathbf{w}_m \|^2 \tag{84}$$

Since $\mathbf{H}_1$ is the subspace $\frac{1}{2} (\mathbf{w}_n + \mathbf{w}_m) = \mathbf{w} \in \mathbf{H}_1$

Thus we can write

$$4 \left\| \mathbf{b} - \frac{1}{2} (\mathbf{w}_n + \mathbf{w}_m) \right\|^2 \geq 4 \inf_{w \in \mathbf{H}_1} \| \mathbf{b} - \mathbf{w} \| = 4\, d^2 \tag{85}$$

Then

$$4 \left\| \mathbf{b} - \frac{1}{2} (\mathbf{w}_n + \mathbf{w}_m) \right\|^2 + \| \mathbf{w}_m - \mathbf{w}_n \|^2 \geq 4\, d^2 + \| \mathbf{w}_m - \mathbf{w}_n \|^2 \tag{86}$$

From (86) relation (84) becomes

$$4\, d^2 + \| \mathbf{w}_m - \mathbf{w}_n \|^2 \leq 2\, \| \mathbf{b} - \mathbf{w}_n \|^2 + 2\, \| \mathbf{b} - \mathbf{w}_m \|^2 \tag{87}$$

One obtains

$$\| \mathbf{w}_m - \mathbf{w}_n \|^2 \leq \frac{1}{2}\, \| \mathbf{b} - \mathbf{w}_n \|^2 + \frac{1}{2}\, \| \mathbf{b} - \mathbf{w}_m \|^2 - d^2 \tag{88}$$

Let $m, n \rightarrow \infty$. The right term of (88) tends to zero according to (81). It follows that the sequence $\{\mathbf{w}_i\}$ is Cauchy and hence $\{\mathbf{w}_i\}$ allows a boundary which belongs to the closed subspace of the complete space.

The limit of a convergent sequence is unique for which $\mathbf{z}$ is defined.

$$\lim_{i \to \infty} \mathbf{w}_i = \mathbf{z} = \mathbf{L}_T \left( \mathbf{G}(\mathbf{y}) \right) \tag{89}$$

ensures the minimum of the norm $\|\mathbf{u}\|$ in $\mathbf{L}^2(0,\ T;\ U)$, hence of the nonlinear system energy.

The continuity condition of the operator $\mathbf{L}_t \left( \mathbf{G}(\mathbf{y}) \right)$ at $[0,\ T]$ implies

$$\mathbf{z} = \lim_{i \to \infty} \mathbf{L}_T \left( \mathbf{G}(\mathbf{y}) \right) \tag{90}$$

Once we obtain $\mathbf{z}$ energy $\mathbf{E}_T^N(0,\ \mathbf{b})$ exists and is unique on $[0,\ T]$, for the class of nonlinear systems under consideration.

Out of relation (79) one obtains

$$\begin{aligned} \left[ \int_0^T \|\mathbf{u}(s)\|^2 \, ds \right] &\le \|\mathbf{b}\|_T + \|\mathbf{L}_T\| \, \|\mathbf{G}(y)\| \le \\ \|\mathbf{b}\|_T &+ \left[ \int_0^T \|\mathbf{G}(y(s))\|_{\mathbf{U}}^2 \, ds \right]^{1/2} \le \\ \|\mathbf{b}\|_T &+ \mathbf{M}_{r,T} \left[ \int_0^T \|\mathbf{G}(y(s))\|_{\mathbf{U}}^2 \, ds \right]^{1/2} \end{aligned} \tag{91}$$

By using restrictions (2.10) and (2.11) provided by

$$\|\mathbf{G}(y(s))\|_{\mathbf{U}} \le \mathbf{M}_{r,T} \ \|\mathbf{y}(\mathbf{s})\|_{\mathbf{T}}\ ,\ \ \|\mathbf{y}\| \le r \tag{92}$$

we get

$$\left[ \int_0^T \|\mathbf{u}(s)\|^2 \, ds \right]^{1/2} \le \|\mathbf{b}\|_T + r\,\mathbf{M}_{r,T} \ \sqrt{T} \tag{93}$$

and

$$\int_0^T \|\mathbf{u}(s)\|^2 \, ds = \mathbf{E}_T^N(0,\ \mathbf{b}) \le \left( \|\mathbf{b}\|_T + r\,\mathbf{M}_{r,T} \ \sqrt{T} \right)^2 \tag{94}$$

respectively.

Similarly we have

$$\int_0^T \|\mathbf{u}(s)\|^2 \, ds = \mathbf{E}_T^N(0,\ \mathbf{b}) \le \left( \|\mathbf{b}\|_T - r\,\mathbf{M}_{r,T} \ \sqrt{T} \right)^2 \tag{95}$$

The existence of a unique solution $\mathbf{E}_T^N(0,\ \mathbf{b})$ on $[0,\ T]$ for the nonlinear system (21) is expressed by satisfying conditions (46).

$$2\,\mathbf{M}_{r,T} \ \sqrt{T} < 1$$

$$\sup_{t \le T} \|\mathbf{L}_t\,\mathbf{u}\|_T \le r \left( 1 - 2\,\mathbf{M}_{r,T} \ \sqrt{T} \right)$$

The first condition results from the convergence of the series $\sum_{m=1}^{\infty} k^{m-1} \| \Phi(0) \|$.

A second condition is obtained from (60) and (78)

$$\sup_{t \leq T} \| \mathbf{L}_t \mathbf{u} \|_T \leq \| \mathbf{b} \|_T \leq r \left( 1 - 2 \mathbf{M}_{r,T} \sqrt{T} \right) \tag{96}$$

The theorem has been demonstrated. □

The use of the theorem 1 and 3 enables the determination of a relation between the energy of the nonlinear $\mathbf{E}_T^N$ system and the linear one $\mathbf{E}_T$.

**Corollary 1.**

We assume that for $T > 0$, the nonlinear term $\mathbf{G}(y)$ satisfies the condition

$$\| \mathbf{G}(0) - \mathbf{G}(b) \|_{\mathbf{U}} \leq \mathbf{M}_{r,T} \| \mathbf{b} \|_T \ , \ \ \| \mathbf{b} \|_T < r \tag{97}$$

If $\mathbf{M}_{r,T} \to 0$M at $r \to 0 \Longrightarrow$

$$\lim_{r \to 0} \mathbf{E}_T(0, \mathbf{b}) = 0 \tag{98}$$

*Proof*

In the topology we have chosen we get

$$\mathbf{M}_{r,T} = \left\| \left( \mathbf{Q}_T^{1/2} \right)^{-1} \mathbf{b} \right\| = \sqrt{\mathbf{E}_T(0, \mathbf{b})} \tag{99}$$

Out of the Theorem 3 one obtains

$$\left( \sqrt{\mathbf{E}_T(0, \mathbf{b})} - r \mathbf{M}_{r,T} \sqrt{T} \right)^2 \leq \mathbf{E}_T(0, \mathbf{b}) \leq \left( \sqrt{\mathbf{E}_T(0, \mathbf{b})} + r \mathbf{M}_{r,T} \sqrt{T} \right)^2 \tag{100}$$

Dealing with the limit, it follows that

$$\lim_{r \to 0} \mathbf{E}_T^N(0, \mathbf{b}) = 0 \tag{101}$$

The corollary has been demonstrated. □

By applying the Theorem 1 relation (100) for $\| \mathbf{b} \|_T < r$, $T > T_0$ becomes

$$\begin{aligned} \sqrt{\mathbf{C}_T} \left\| \left( \mathbf{Q}_T^{1/2} \right)^{-1} \mathbf{b} \right\| + \mathbf{M}_{r,T} \sqrt{T} &\leq \mathbf{E}_T^N(0, \mathbf{b}) \\ &\leq \left( \sqrt{\mathbf{C}_T} \left\| \left( \mathbf{Q}_T^{1/2} \right)^{-1} \mathbf{b} \right\| + r \mathbf{M}_{r,T} \sqrt{T} \right)^2 \end{aligned} \tag{102}$$

where operator $\mathbf{Q}$ is defined in (7).

## 4. Conclusions

This study is meant to determine the qualitative and quantitative characteristics referring to the minimum energy of nonlinear dynamic systems.

In the class of nonlinear systems analyzed there have been satisfied the Lipschitz conditions. The mathematical model proposed makes use of functional analytical elements applied to the Banach and Hilbert space. Thus one has demonstrated the existence and uniqueness of the solution in issues of minimizing the energy for the nonlinear case.

The topology introduced has been defined with the controllability operator.

The properties induced by the topology considered in space $\mathbf{L}^2(0, T; \mathbf{U})$ determines the evaluation of the domain of variation for the minimum energy in the interval of time specified $[0, T]$.

On the basis of the results obtained one acquires the relation between the minimum energy for nonlinear systems associated with them.

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