Totally Singular Control for Systems with Parameters

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Abstract

Abstract. Here we study problems of singular optimal control for which the index of performance, the differential constraints and the final conditions contain parameters. We determinate the trajectory of the neighboring extremal for the initial point perturbed and perturbed final manifold. This allows to obtain the second variation in the singular case. The sufficient conditions of minimum are a consequence of the non-negativity condition for the second variation.

1. Formulation of the Problem

We consider the problem of optimality containing parameters with a constant final time and the vector of state $x = (x_1, x_2, \ldots x_n)^T$.

We determine the vector of control $u = (u_1, u_2, \ldots u_m)^T$ and the parameter $p = (p_1, p_2, \ldots p_r)^T$ which minimize the performance index

$$J = \Phi(x, p) + \int_{t_0}^{t_f} L(t, x, u, p)dt \tag{1}$$

with respect to the differential constraints

$$\dot{x} = f(t, x, u, p)\,, \tag{2}$$

which satisfy the initial conditions

$$t_0 = 0 \qquad\qquad x_0 = x(0)\,, \tag{3}$$

and the final conditions

$$t_f = \text{given} \qquad \Psi(x_f, p) = 0\,, \tag{4}$$

where Φ and L are scalars and Ψ is a vector of dimension $s \times 1$ defined by

$$\Psi(x_f, p) = \begin{pmatrix} \eta(x_f) \\ \theta(p) \end{pmatrix}\,. \tag{5}$$

If $\eta(x_f)$ and $\theta(p)$ are vectors of dimension $l \times 1$ and $q \times 1$, respectively, for $l \leq n$, $q \leq r$, then $s \leq n + r$.

Let $\mathcal{C}_p$ be the class of problems of optimality with parameters defined in (1-5).

Denote by $\mathcal{C}_{t_f}$ the class obtained by particularization $p = t_f$, $\mathcal{C}_{t_f} \subset \mathcal{C}_p$.

With the transformation

$$\tau = t/t_f, \tag{6}$$

we can write

$$\mathcal{C}_{t_f} = \left\{\min J \mid \tau_f = 1,\ \bar{L} = t_f L,\ \bar{f} = t_f f,\ p = t_f\right\}\,. \tag{7}$$

For $\mathcal{C}_{t_f}$, the initial conditions are written

$$\tau_0 = 0, \qquad x_0 = x(0)\,, \tag{8}$$

and the final conditions become

$$\tau_f = 1, \qquad \Psi(x_f, t_f) = 0\,. \tag{9}$$

Thus, the problem with free final time becomes a problem with constant final time $\tau = 1$.

2. First Order Variation

The extended functional becomes

$$J' = G\left(x_f, \nu, p\right) + \int_{t_0}^{t_f} \left[H\left(t, x, u, \lambda, p\right) - \lambda^T \dot{x}\right] dt\,, \tag{10}$$

where the function of the final values and the variational Hamiltonian are defined by

$$G = \Phi + \nu^T \Psi\,, \tag{11}$$

$$H = L + \lambda^T f\,. \tag{12}$$

Along the admissible trajectory $\delta x_0 = 0$, we obtain

$$\delta J' = \left(G_{x_f} - \lambda_f^T\right) \delta x_f + \Psi^T \delta \nu + \left(G_p + \int_{t_0}^{t_f} H_p \, dt\right) \delta p$$
$$+ \int_{t_0}^{t_f} \left[\left(H_x + \dot{\lambda}^T\right) \delta x + H_u \, \delta u + \left(f^T - \dot{x}^T\right) \delta \lambda\right] d\,t \,. \tag{13}$$

Along the admissible trajectory of comparison, where $\delta p \neq 0$, the necessary conditions of minimum resulted by vanishing the first variation can be written

$$\dot{x} = f, \qquad \dot{\lambda} = -H_x^T \,, \tag{14a}$$

$$H_u^T = 0, \qquad G_p + \int_{t_0}^{t_f} H_p \, dt = 0 \,, \tag{14b}$$

$$t_0 = \text{given}, \qquad x_0 = \text{given} \,, \tag{14c}$$

$$t_f = \text{given}, \qquad \Psi = 0, \qquad \lambda_f = G_{x_f}^T \,. \tag{14d}$$

3. Second Order Variation

The expression of the second variation:

$$\delta^2 \mathbf{J}' = \begin{pmatrix} \delta x_f^T & \delta p^T \end{pmatrix} \begin{pmatrix} G_{x_f x_f} & G_{x_f p} \\ G_{p x_f} & G_{pp} \end{pmatrix} \begin{pmatrix} \delta x_f \\ \delta p \end{pmatrix}$$
$$+ \int_{t_0}^{t_f} \begin{pmatrix} \delta x^T & \delta u^T & \delta p^T \end{pmatrix} \begin{pmatrix} H_{xx} & H_{xu} & H_{xp} \\ H_{ux} & H_{uu} & H_{up} \\ H_{px} & H_{pu} & H_{pp} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta u \\ \delta p \end{pmatrix} dt \,. \tag{15}$$

Using the function

$$\mu_i = -\int_{t_0}^{t} H_{p_i} \, d\tau \qquad i = 1, 2, ..., r \,, \tag{16}$$

equation (16b) is transformed into the differential equation

$$\dot{\mu}_i = -H_p^T \,, \tag{17}$$

with the initial condition

$$\mu(t_0) = \mu_0 = 0\,, \tag{18}$$

and the final condition

$$\mu(t_f) = G_p^T\,. \tag{19}$$

The properties (14a-14d) (defining the extremal trajectory) become

$$\dot{x} = f(t,\,x,\,u,\,p), \qquad \dot{\lambda} = -H_x(t,\,x,\,u,\,\lambda,\,p)\,, \tag{20a}$$

$$\dot{\mu} = H_x(t,\,x,\,u,\,\lambda,\,p), \qquad 0 = H_u^T(t,\,x,\,u,\,\lambda,\,p)\,, \tag{20b}$$

$$t_0 = \text{given}, \qquad x_0 = \text{given}, \qquad \mu_0 = 0\,, \tag{20c}$$

$$t_f = \text{given}, \quad \Psi(x_f, p) = 0, \quad \lambda_f = G_{x_f}^T(x_f,\,\nu,\,p), \quad \mu_f = G_p^T(x_f,\,\nu,\,p)\,. \tag{20d}$$

The trajectory of the neighboring extremal that corresponds to the perturbed initial point $\delta x_0 = 0$ and perturbed final constraints is obtained by the variation of the equations (20). In the case of the totally singular control we cannot use the variation of H_u because $H_{uu} = 0$. Thus, it is necessary to develop a method to determine the variation of the command along the extremal.

4. Totally Singular Control

Consider the controlled systems of the form

$$\dot{x} = f_0(t,\,x) + f_1(t,\,x)u \qquad t \in [t_0,\,t_f]\,, \tag{21}$$

with

$$x(t_0) = x_0\,, \tag{22}$$

$$\Psi(x(t_f)) = 0\,. \tag{23}$$

For the singular control exists a subset of commands $u(t)$ where the Hamiltonian is stationary. Hence we have

$$H_u(t,\,x,\,u,\,\lambda) \equiv 0 \quad \forall\, t \in [t_0, t_f]\,. \tag{24}$$

5. Neighboring Extremal in the Totally Singular Control Case

In the totally singular control case, H_{uu} is the null matrix. The equation $H_u = 0$ of the nonsingular case can no longer be utilized. This condition is substituted by

$$\frac{d^{2k} H_u}{d\,t^{2k}} = H_u^{(2k)} = 0\,, \tag{25}$$

where k is the smallest natural number, such that

$$\frac{\partial}{\partial u}\left(H_u^{(2k)}\right) \neq 0\,. \tag{26}$$

The equations of the neighboring extremal with $\delta x_0 \neq 0$ are obtained using the variation of the equations (20) and (25) which substitute the variation of $H_u = 0$.

The expressions of the variations become

$$\delta \dot{x} = f_x\,\delta x + f_u\,\delta u + f_p\,\delta p\,, \tag{27a}$$

$$\delta \dot{\lambda} = -H_{xx}\,\delta x - H_{xu}\,\delta u - f_x^T\,\delta\lambda - H_{xp}\,\delta p\,, \tag{27b}$$

$$\delta \dot{\mu} = -H_{px}\,\delta x - H_{pu}\,\delta u - f_p^T\,\delta\lambda - H_{pp}\,\delta p\,, \tag{27c}$$

$$\left(H_u^{(2k)}\right)_x\delta x + \left(H_u^{(2k)}\right)_u\delta u + \left(H_u^{(2k)}\right)_\lambda\delta\lambda + \left(H_u^{(2k)}\right)_p\delta p = 0\,, \tag{27d}$$

From (27d), we get the control variation on the neighboring extremal

$$\delta u = -\left[\left(H_u^{(2k)}\right)_u\right]^{-1}\left[\left(H_x^{(2k)}\right)_x\delta u + \left(H_u^{(2k)}\right)_\lambda\delta\lambda + \left(H_u^{(2k)}\right)_p\delta p\right]\,. \tag{38}$$

Replacing (28) in (27) the equations of the neighboring extremal can be written

$$\delta \dot{x} = A_1\delta x + B_1\delta\lambda + C_1\delta p\,, \tag{29a}$$

$$\delta \dot{\lambda} = A_2\delta x + B_2\delta\lambda + C_2\delta p\,, \tag{29b}$$

$$\delta \dot{\mu} = A_2\delta x + B_3\delta\lambda + C_3\delta\,, p \tag{29c}$$

where

$$A_1 = f_x - f_u\left[\left(H_u^{(2k)}\right)_u\right]^{-1}\left(H_u^{(2k)}\right)_x\,, \tag{30a}$$

$$B_1 = -f_u\left[\left(H_u^{(2k)}\right)_u\right]^{-1}\left(H_u^{(2k)}\right)_\lambda\,, \tag{30b}$$

$$C_1 = -f_u\left[\left(H_u^{(2k)}\right)_u\right]^{-1}\left(H_u^{(2k)}\right)_p + f_p\,, \tag{30c}$$

$$A_2 = H_{xx} - H_{xu}\left[\left(H_u^{(2k)}\right)_u\right]^{-1}\left(H_u^{(2k)}\right)_x\,, \tag{30d}$$

$$B_2 = -H_{xu}\left[\left(H_u^{(2k)}\right)_u\right]^{-1}\left(H_u^{(2k)}\right)_\lambda - f_x^T\,, \tag{30e}$$

$$C_2 = -H_{xu}\left[\left(H_u^{(2k)}\right)_u\right]^{-1}\left(H_u^{(2k)}\right)_p - H_{pp}\,, \tag{30f}$$

$$A_3 = H_{px} - H_{pu}\left[\left(H_u^{(2k)}\right)_u\right]^{-1}\left(H_u^{(2k)}\right)_x\,, \tag{30g}$$

$$B_3 = -H_{pu}\left[\left(H_u^{(2k)}\right)_u\right]^{-1}\left(H_u^{(2k)}\right)_\lambda + f_p^T\,, \tag{30h}$$

$$C_3 = -H_{pu}\left[\left(H_u^{(2k)}\right)_u\right]^{-1}\left(H_u^{(2k)}\right)_p + H_{pp}\,. \tag{30i}$$

The initial conditions for the neighboring extremal are given by

$$t_0 = 0\,, \qquad \delta x_0 = \text{given}\,, \qquad \delta\mu_0 = 0\,, \tag{31}$$

and the final conditions are obtained by the variation of conditions (20d).

Thus, we have

$$\delta\lambda_f = G_{x_f x_f}\delta x_f + \Psi_{x_f}^T\,\delta\nu + G_{x_f\,p}\,\delta p\,, \tag{32a}$$

$$\delta\Psi = \Psi_{px_f}\,\delta x_f + \Psi_p\,\delta p\,, \tag{32b}$$

$$\delta\mu_f = G_{p\,x_f}\,\delta x_f + \Psi_p^T\,\delta\nu + G_{p\,p}\,\delta p\,, \tag{32c}$$

where

$$\delta\Psi = 0\,. \tag{33}$$

Then, the variations of $\delta\lambda,\ \delta\Psi,\ \delta\mu$

$$\delta\lambda = p_1\,\delta x_f + Q_1\,\delta\nu + R_1\,\delta p\,, \tag{34a}$$

$$0 = p_1\,\delta x_f + Q_2\,\delta\nu + R_2\,\delta p\,, \tag{34b}$$

$$\delta\mu = p_3\,\delta x_f + Q_3\,\delta\nu + R_3\,\delta p\,, \tag{34c}$$

where the final conditions of system (34), are obtained by identification with (32)

$$(P_1)_f = G_{x_f x_f} \qquad (Q_1)_f = \Psi_{x_f}^T \qquad (R_1)_f = G_{x_f\,p}\,, \tag{35a}$$

$$(P_2)_f = \Psi_{x_f} \qquad (Q_2)_f = 0 \qquad (R_2)_f = \Psi_p\,, \tag{35b}$$

$$(P_3)_f = G_{p\,x_f} \qquad (Q_3)_f = \Psi_p^T \qquad (R_3)_f = G_{pp}\,. \tag{35c}$$

In the following, we determinate the differential equations with the unknowns $P_i,\ Q_i,\ R_i\ \ (i = 1, 2, 3)$ which satisfy the final conditions (35). In our developments, we consider $\delta\dot\nu = \delta\dot p = 0\,.$

6. Differential System for $P_i,\ Q_i,\ R_i$

By the derivation of the equation (34a) and by the substitution of $\delta\dot x$ given by (29a), using the expression of $\delta\lambda$ from (34a), the identification with (29b) of the coefficients of $\delta x,\ \delta\nu,\ \delta p$ one obtains a differential system $(\sum)$ in $P_1,\ Q_1,\ R_1$.

The differential system $(\sum)$, with the conditions at the limit (35) we determinate the coefficients $P_i,\ Q_i,\ R_i\ \ (i = 1, 2, 3)$ of the variations $\delta\lambda,\ \delta\Psi,\ \delta\mu$.

7. Extremal Neighboring Trajectory

The equations

$$0 = P_2\delta x + Q_2\delta\nu + R_2\delta p\,, \tag{36a}$$

$$\delta\mu = P_3\delta x + Q_3\delta\nu + R_3\delta p\,, \tag{36b}$$

$$\delta x_0 = \text{given}, \qquad \delta\mu_0 = 0\,, \tag{36c}$$

solved with respect to the initial point, simultaneous for $\delta\nu$ and δp, have the solution

$$\begin{pmatrix} \delta\nu \\ \delta p \end{pmatrix} = -V_0^{-1}U_0\,\delta x_0, \tag{37}$$

where

$$V = \begin{pmatrix} Q_2 & R_2 \\ Q_3 & R_3 \end{pmatrix}, \qquad U = \begin{pmatrix} P_2 \\ P_3 \end{pmatrix}. \tag{38}$$

Starting from the expression $\delta\lambda$ of given by (34a) and using (37), we have

$$\delta\lambda_0 = \left[(P_1)_0 - (Q_1\ \ R_1)_0\,V_0^{-1}U_0\right]\delta x_0 = K_0\delta x_0\,, \tag{40}$$

where

$$K(t) = P_1 - (Q_1\ \ R_1)\,V^{-1}U\,. \tag{51}$$

If δx_0 is given, then $\delta\lambda_0$ and δp are calculated from the equations (39) and (37). As $\delta\mu_0 = 0$, with the initial conditions $(\delta x_0,\, \delta\lambda_0, \delta\mu_0)$, the variations $(\delta x,\, \delta\lambda, \delta\mu)$ are obtained by integrating the system (29). Knowing the variations $\delta x,\, \delta\lambda, \delta p$, by (28) and (34a) we can determinate the control perturbation on the neighboring extremal,

$$\delta u = -\left[\left(H_u^{(2k)}\right)_u\right]^{-1}\left\{\left[\left(H_u^{(2k)}\right)_x\left(H_u^{(2k)}\right)_\lambda P_1\right]\delta x\right\} - \left[\left(H_u^{(2k)}\right)_u\right]^{-1} \left\{\left[\left(H_u^{(2k)}\right)_\lambda Q_1\right]\delta\nu + \left[\left(H_u^{(2k)}\right)_\lambda R_1 + \left(H_u^{(2k)}\right)_p\right]\delta p\right\}. \tag{52}$$

Using (37) and (39), we obtain the following results proposition

Proposition 7.1. *If the matrices* $\mathbf{V}_0^{-1}U_0$ *and* $\mathbf{K}_0$ *are finite, then any admissible trajectory compared* $\zeta(t)$*, does not belong to the class of neighboring extremals.*

Proof. Consider the set of compared admissible trajectories

$$\zeta(t) = \{\, x(t) \mid \delta x(t_0) = 0 \,\}\,. \tag{42}$$

We consider $V_0^{-1}U_0$ to be finite. Then, for $x(t) \in \zeta(t)$, from (37) and (39) we have

$$\delta\nu = 0\,, \tag{43a}$$

$$\delta p = 0\,, \tag{43b}$$

$$\delta\lambda_0 = 0\,. \tag{43c}$$

For $\delta p = 0$, the variational equations (29a) and (29b) become:

$$\delta\dot{x} = A_1\delta x + B_1\delta\lambda\,, \tag{44a}$$

$$\delta\dot{\lambda} = A_2\delta x + B_2\delta\lambda\,. \tag{44b}$$

The solution of the system (44) for the initial conditions $\delta x_0 = 0$ and $\delta\lambda_0 = 0$ is $\delta x = 0$ and $\delta\lambda = 0$. Then, from (38) we obtain eighter $\delta u = 0$ or $\zeta(t) \notin \Gamma$. □

Proposition 7.2. *If* $\mathbf{V}_0^{-1}U_0$ *is infinite, then any admissible trajectory can be a neighboring extremal.*

Proof. Along the admissible trajectory of comparison $\delta x_0 = 0$. Then we obtain

$$\delta\lambda_0 = K_0\delta x_0 \neq 0\,. \tag{45}$$

Therefore, $\delta u \neq 0$ or $\zeta(t) \in \Gamma$. □

8. Sufficient Minimum Conditions in the Totally Singular Case

Substituting the control of the perturbation along the neighboring extremal (41) in the expression of the second variation for $H_{uu} = 0$, we obtain

$$\delta^2 J' = \begin{pmatrix} \delta x_f^T & \delta p^T \end{pmatrix} \begin{pmatrix} G_{x_f x_f} & G_{x_f p} \\ G_{p x_f} & G_{pp} \end{pmatrix} \begin{pmatrix} \delta x_f \\ \delta p \end{pmatrix}$$
$$+ \int_{t_0}^{t_f} \begin{pmatrix} \delta x^T & \delta u^T & \delta p^T \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12}^T & 0 & S_{23}^T \\ S_{13}^T & S_{23} & S_{33} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta u \\ \delta p \end{pmatrix} dt\,, \tag{46}$$

where

$$S_{11} = H_{xx} - 2\,H_{xu}\left[\left(H_u^{(2k)}\right)_u\right]^{-1} D_1\,, \tag{47a}$$

$$S_{12} = -H_{xu}\left[\left(H_u^{(2k)}\right)_u\right]^{-1} D_2\,, \tag{47b}$$

$$S_{13} = H_{xp} - H_{xu}\left[\left(H_u^{(2k)}\right)_u\right]^{-1} D_3 - D_1^T \left\{\left[\left(H_u^{(2k)}\right)_u\right]^{-1}\right\}^T H_{up}\,, \tag{47c}$$

$$S_{23} = -H_{pu}\left[\left(H_u^{(2k)}\right)_u\right]^{-1} D_2\,, \tag{47d}$$

$$S_{33} = H_{pp} - 2\,H_{pu}\left[\left(H_u^{(2k)}\right)_u\right]^{-1} D_3\,, \tag{47e}$$

$$D_1 = \left[\left(H_u^{(2k)}\right)_x + \left(H_u^{(2k)}\right)_\lambda p_1\right]\,, \tag{47f}$$

$$D_2 = \left[\left(H_u^{(2k)}\right)_\lambda Q_1\right]\,, \tag{47g}$$

$$D_3 = \left[\left(H_u^{(2k)}\right)_\lambda R_1 + \left(H_u^{(2k)}\right)_p\right]\,. \tag{47h}$$

Two cases are possible:

Case 1. The matrices $V_0^{-1}U_0$ and K_0 are finite. In this case, as the variations $\delta\nu$, δp and δx are null for any $t \in [t_0, t_f]$, from (46) we have

$$\delta^2 J' = 0\,. \tag{48}$$

Case 2. The matrix $V_0^{-1}U_0$ is infinite. As $\delta x_0 = 0$ along the admissible trajectory of comparison, we can obtain a finite $\delta\lambda_0$ different from zero and we can have a neighboring extremal trajectory that can also be an admissible trajectory.

The sufficient condition presented in Theorem 9.1 implies the sufficient conditions of nonnegativity for the quadratic functionals.

Theorem 8.2. *The sufficient condition $\delta^2 J' \geq 0$ for $\Psi(x_f, p) = 0$, imposes the existence of a symmetrical matrix $M(t)$ positive semidefined and of a symmetrical matrix positive semidefined N, such that*

$$\delta^2 J' \geq \int_{t_0}^{t_f} \left(\frac{1}{2}x^T M_{11} x + \alpha^T M_{21} x + \frac{1}{2}\alpha^T M_{22}\alpha\right) dt + \frac{1}{2} y^T N y(t)\,. \tag{49}$$

The expression (46) of the second variation can be rewritten as

$$\delta^2 J' = \begin{pmatrix} \delta x_f & \delta p \end{pmatrix} \begin{pmatrix} G_{x_f x_f} & G_{x_f p} \\ G_{p x_f} & G_{pp} \end{pmatrix} \begin{pmatrix} \delta x_f \\ \delta p \end{pmatrix} +$$

$$\int_{t_0}^{t_f} \begin{pmatrix} \delta x^T & \left(-V_0^{-1}U_0\delta x_0\right)^T \end{pmatrix} \begin{pmatrix} S_{11} & \vdots & \overline{S}_{12} \\ \cdots & \cdots & \cdots \\ \overline{S}_{12}^T & \vdots & \overline{S}_{22} \end{pmatrix} \begin{pmatrix} \delta x \\ \cdots \\ -V_0^{-1}U_0\delta x_0 \end{pmatrix} dt\,, \tag{50}$$

where

$$\overline{S}_{12} = \begin{pmatrix} S_{12} & \vdots & S_{13} \end{pmatrix}, \tag{51a}$$

$$\overline{S}_{22} = \begin{pmatrix} 0 & \vdots & S_{23} \\ \cdots & \cdots & \cdots \\ S_{23} & \vdots & S_{33} \end{pmatrix}, \tag{52b}$$

If we take

$$M(t) = \overline{S}(t) = \begin{pmatrix} S_{11} & \vdots & \overline{S}_{12} \\ \cdots & \cdots & \cdots \\ \overline{S}_{12}^{T} & \vdots & \overline{S}_{22} \end{pmatrix}, \tag{53}$$

and

$$N = \begin{pmatrix} G_{x_f x_f} & G_{x_f p} \\ G_{p x_f} & G_{pp} \end{pmatrix}, \tag{54}$$

then the using of the Theorem 8.1 reduces to determining of the conditions of nonnegativity of the symmetrical matrices $M(t)$ and N.

9. Conclusions

The current study refers to the singular total case in which the second variation cannot be strongly positive. This confirms that the Riccati differential matricial equation attached to the non-singular problem cannot be used. In literature [4 - 9] the necessary and sufficient conditions of non-negativity of the second variation are represented by a set of differential and algebraical equations. Our method analyzes the possibility when the neighboring extremal to can be the admissible trajectory and it determines the variation of the command along the extremal. Thus, if the singular arcs defined by means of abnormal curves belong to the extremals, then the normal extremal curves do not admit solutions with the variation of the state identically null in any time interval. This propriety demonstrates the uniqueness of the Jacobi solution along the normal extremal. The previous researches assume the construction of a material function of time that is not usual in the absence of a hypothesis. The mathematical model elaborated here determines the conditions of nonnegativity of the second variation, resulted from the variation of the command along the extremal, representing the sufficient conditions of minimum for the class of the problems of optimum with parameter.

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