Sufficient Conditions for Existence the Solution of Linear Two-Point Boundary Problem in Minimization of Quadratic Functional

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Abstract

The quadratic functional minimization with differential restrictions represented by the command linear systems is considered. The optimal solution determination implies the solving of a linear problem with two points boundary values. The proposed method implies the construction of a fundamental solution $\mathbf{S}(t)$ - a $n \times n$ matrix- and of a vector $\mathbf{h}(t)$ defining an adjoint variable $\lambda(t)$ depending of the state variable $\mathbf{x}(t)$. From the extremum necessary conditions it is obtained the Ricatti matrix differential equation having the $\mathbf{S}(t)$ as unknown fundamental solution is obtained. The paper analyzes the existence of the Ricatti equation solution $\mathbf{S}(t)$ and establishes as the optimal solution of the proposed optimum problem. Also a superior limit of the minimum for the considered quadratic functionals class are evaluated.

1 Introduction

In the control theory a special importance is accorded to the quadratic linear problem. The interest is justified by the great number of its practical applications.

A representative model is offered by the linear regulator problem corresponding to the quadratic functionals minimization, with differential restrictions, defined by linear command systems [2], [3], [4],[5].

Detailing the Bellman equation, the Ricatti differential equation associated to the proposed optimum problem is obtained.

The optimal control feedback and the minimum value of the performance index is expressed as a function of the Riccati [2], [4], [5]. Therefore, determining the existence conditions for the solution of Riccati equation becomes a necessity.

Also, taking into account the hypothesis of the nilpotent structure of the bilinear systems the optimal control for the quadratic functionals class [7], [8], [9] is obtained.

The control on the neighbour extremal utilizes the transition matrices and their symplectic properties. The admissible optimal neighbour trajectory is obtained by the integration of the variational Hamiltonian system with boundary conditions obtained by the annulation of the extremised functional. This approach is a linear problem with two point boundary values. The existent results in the quadratic linear problem can be extended to differential restrictions having the form of command systems with a free, perturbing term. The present study is based on this approach.

2 The optimal control problem

Let's consider the following controlled linear differential systems, of order n:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u} + \mathbf{F}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathbf{R}^n, \quad t \in [0, t_f] \qquad (1)$$

The elements of the matrices $\mathbf{A}$ and $\mathbf{B}$ and the components of the vector $\mathbf{F}(t)$ are continuous real functions defined within $[0, t_f]$.

$\mathbf{E} = \mathbf{R}^n$ is the state space and $\mathbf{U} = \mathbf{R}^m$ represents the parameters space. $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{B}$, $\mathbf{x}$, $\mathbf{F}$ and $\mathbf{u}$ are, of the $n \times n$, n, n, m dimension, respectively.

Let's assume that the quadratic performance index is expressed by the following functionals:

$$\mathbf{J}_{t_f} = \frac{1}{2}\int_0^{t_f} \left[\mathbf{x}^T\mathbf{Q}(t)\mathbf{x} + \mathbf{u}^T\mathbf{R}(t)\mathbf{u}\right] dt + \frac{1}{2}\mathbf{x}^T(t_f)\Phi(t_f)\mathbf{x}(t_f) \qquad (2)$$

where $\mathbf{Q}$, $\mathbf{\Phi}$ are $n \times n$ symmetrical nonnegative matrices and $\mathbf{R}$ is a $m \times m$ positive defined matrix.

The proposed optimum problem is equivalent with the determination of the control vector $\mathbf{u} \in \mathbf{U}$ which minimizes the performance index (2) under the restrictions (1).

Then, the Hamiltonian $\mathbf{H}$ can be written:

$$\begin{aligned}\mathbf{H}(x,\lambda,u,t) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}(t)\mathbf{x} + \frac{1}{2}\mathbf{u}^T\mathbf{R}(t)\mathbf{u} + \\ \lambda^T\left[\mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u} + \mathbf{F}(t)\right]\end{aligned} \tag{3}$$

and the optimal control $\mathbf{u}^*$ is obtained from:

$$\mathbf{H}_u(x,\lambda,u,t) = 0 \tag{4}$$

where

$$\mathbf{u}^* = -\mathbf{R}^{-1}\mathbf{B}^T\lambda \tag{5}$$

Changing $\mathbf{u}^*$ in (3), the optimal Hamiltonian $\mathbf{H}^*$ can be written:

$$\mathbf{H}^* = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \lambda^T\mathbf{A}\mathbf{x} - \frac{1}{2}\lambda^T\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\lambda + \lambda^T\mathbf{F} \tag{6}$$

The determination of the optimal solution results from the integration of the Hamiltonian system:

$$\begin{aligned}\dot{\mathbf{x}} &= \frac{\partial \mathbf{H}^*}{\partial \lambda}\\ \dot{\lambda} &= -\frac{\partial \mathbf{H}^*}{\partial x}\end{aligned} \tag{7}$$

with the boundary-conditions

$$\begin{aligned}&\mathbf{x}(0) = \mathbf{x}_0 && \text{a)}\\ &\lambda(t_f) = \mathbf{\Phi}(t_f)\mathbf{x}(t_f) && \text{b)}\end{aligned} \tag{8}$$

Solving of the equation (7) under the above conditions (8) defines the two-point linear boundary value problem.

3 Solving method for two-point linear boundary value problems

Explicating (7) it appears that

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\lambda + \mathbf{F}\\ \dot{\lambda} &= -\mathbf{Q}\mathbf{x} - \mathbf{A}^T\lambda\end{aligned} \tag{9}$$

Using the symplectic properties of the transition matrices, the case $\mathbf{F} = 0$ has been discussed in [6] and [11].

We aim to build a fundamental solution for solving the linear two-point boundary value problem represented by the inhomogeneous differential system (9) within the boundary conditions (8).

Let's consider an $\mathbf{S}(t)$ square matrix of order n and an $\mathbf{h}(t)$ vector of dimension n, which will be determinated so that the solution $\mathbf{x}(t)$ and $\lambda(t)$ of the system (9) with the final solution 8(b) satisfies the relation

$$\lambda(t) = \mathbf{S}(t)\mathbf{x}(t) + \mathbf{h}(t) \tag{10}$$

The differential equations for $\mathbf{S}(t)$ and $\mathbf{h}(t)$ are chosen so that to have

$$\frac{d}{dt}\Big[\mathbf{S}(t)\mathbf{x}(t) + \mathbf{h}(t) - \lambda(t)\Big] = 0 \tag{11}$$

for any solution of the equation (9).

From(11) it follows

$$\dot{\mathbf{S}}\mathbf{x} + \mathbf{S}\dot{\mathbf{x}} + \dot{\mathbf{h}} - \dot{\lambda} = 0 \tag{12}$$

Replacing the adjoint variable (10) in (9), it appears that

$$\begin{aligned} &\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S})\mathbf{x} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{h} + \mathbf{F} && \text{a)} \\ &\dot{\lambda} = -(\mathbf{Q} + \mathbf{A}^T\mathbf{S})\mathbf{x} - \mathbf{A}^T\mathbf{h} && \text{b)} \end{aligned} \tag{13}$$

Considering (13) the differential system (12) becomes

$$\begin{gathered} \big[\dot{\mathbf{S}} + \mathbf{Q} + \mathbf{S}\mathbf{A} + \mathbf{A}^T\mathbf{S} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S}\big]\mathbf{x} + \dot{\mathbf{h}} + \\ (-\mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S} + \mathbf{A}^T)\mathbf{h} + \mathbf{S}\mathbf{F} = 0 \end{gathered} \tag{14}$$

Relation (14) is satisfied for any $\mathbf{x}$ if $\mathbf{P}(t)$ and $\mathbf{h}(t)$ are determined such that we get

$$\dot{\mathbf{S}} + \mathbf{Q} + \mathbf{S}\mathbf{A} + \mathbf{A}^T\mathbf{S} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S} = 0 \tag{15}$$

$$\mathbf{S}(t_f) = \mathbf{\Phi}(t_f) = \mathbf{\Phi}_f \tag{16}$$

respectively

$$\dot{\mathbf{h}} + (-\mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S} + \mathbf{A}^T)\mathbf{h} + \mathbf{S}\mathbf{F} = 0 \tag{17}$$

$$\mathbf{h}(t_f) = 0 \tag{18}$$

The boundary conditions (16) and (18) result from (10) and (8b).

4 Analyze to determine the existence of a solution differential equation

Utilizing the H.G.Moyer's results [1], the sufficient conditions for the existence of a solution of the Riccati matrix differential equation (15) are established.

The sufficient conditions for the existence of $\mathbf{P}(t)$ where $t \in [0, t_f]$ satisfying equation

$$-\dot{\mathbf{S}} = \mathbf{Q} + \mathbf{SA} + \mathbf{A}^T\mathbf{S} - (\mathbf{C} + \mathbf{B}^T\mathbf{S})\mathbf{R}^{-1}(\mathbf{C} + \mathbf{B}^T\mathbf{S}) \tag{19}$$

for which

$$\mathbf{S}(t_f) = \mathbf{\Phi}_f \tag{20}$$

are

$$\mathbf{Q} - \mathbf{C}^T\mathbf{R}^{-1}\mathbf{C} \geq 0 \quad \forall t \in [0, t_f] \tag{21}$$

$$\mathbf{R}^{-1} > 0 \quad \forall t \in [0, t_f] \tag{22}$$

$$\mathbf{\Phi}_f \geq 0 \tag{23}$$

By refining the equation (19) using the notations

$$\bar{\mathbf{Q}} = \mathbf{Q} - \mathbf{C}^T\mathbf{R}^{-1}\mathbf{C} \tag{24}$$

$$\bar{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{C} \tag{25}$$

it is obtained the equation (15); therefore the problem of the existence of a solution for equation (19) is reduced to finding of a solution for the equation (15).

The sufficient condition for the existence of $\mathbf{S}(t)$ where $t \in [0, t_f]$ satisfying the equations

$$-\dot{\mathbf{S}} = \mathbf{Q} + \mathbf{SA} + \mathbf{A}^T\mathbf{S} - \mathbf{SBR}^{-1}\mathbf{B}^T\mathbf{S} \tag{26}$$

$$\mathbf{S}(t_f) = \mathbf{\Phi}_f \tag{27}$$

is the existence of an $n \times n$ symmetric matrix $\mathbf{P}(t)$, having time continuous differentiable functions defined within $t \in [0, t_f]$ as elements, such that

$$\mathbf{B}^T\mathbf{P} = 0 \quad \forall t \in [0, t_f] \tag{28}$$

$$\dot{\mathbf{P}} + \mathbf{Q} + \mathbf{PA} + \mathbf{A}^T\mathbf{P} = \mathbf{M}(t) \geq 0 \quad \forall t \in [0, t_f] \tag{29}$$

$$\mathbf{\Phi}_f - \mathbf{P}(t_f) = \mathbf{G}_f \geq 0 \tag{30}$$

Proof. Let's consider

$$\bar{\mathbf{P}}(t) + \bar{\mathbf{S}}(t) = \mathbf{P}(t) \tag{31}$$

where $\bar{\mathbf{P}}(t)$ and $\bar{\mathbf{S}}(t)$ are symmetrical matrices. According to the hypothesis, a symmetrical $\mathbf{P}(t)$ exists satisfying (28), (29), and (30).

We have

$$-\dot{\mathbf{P}} = \mathbf{Q} + \mathbf{PA} + \mathbf{A}^T\mathbf{P} - \mathbf{M} \tag{32}$$

Utilizing the hypothesis (28) we rewrite (32) as

$$-\dot{\mathbf{P}} = \mathbf{Q} + \mathbf{PA} + \mathbf{A}^T\mathbf{P} - \mathbf{M} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} \tag{33}$$

Substituting the value of $\mathbf{P}$ from (31) in (33) we get

$$\begin{gathered}
-\dot{\bar{\mathbf{P}}} - \dot{\bar{\mathbf{S}}} = \mathbf{Q} + \mathbf{A}^T(\bar{\mathbf{P}} + \bar{\mathbf{S}}) + (\bar{\mathbf{P}} + \bar{\mathbf{S}})\mathbf{A} - \mathbf{M} - \\
(\bar{\mathbf{P}} + \bar{\mathbf{S}})\mathbf{BR}^{-1}\mathbf{B}^T(\bar{\mathbf{P}} + \bar{\mathbf{S}}) = \mathbf{Q} + \mathbf{A}^T(\bar{\mathbf{P}} + \bar{\mathbf{S}}) + \\
(\bar{\mathbf{P}} + \bar{\mathbf{S}})\mathbf{A} - \mathbf{M} - \bar{\mathbf{S}}\mathbf{BR}^{-1}\mathbf{B}^T\bar{\mathbf{S}} - \bar{\mathbf{S}}\mathbf{BR}^{-1}\mathbf{B}^T\bar{\mathbf{P}} - \\
\bar{\mathbf{P}}\mathbf{BR}^{-1}\mathbf{B}^T\bar{\mathbf{S}} - \bar{\mathbf{P}}\mathbf{BR}^{-1}\mathbf{B}^T\bar{\mathbf{P}}
\end{gathered} \tag{34}$$

Further substituting

$$\bar{\mathbf{S}} = \mathbf{P} - \bar{\mathbf{P}} \tag{35}$$

and considering (28) the equation (34) becomes

$$\begin{gathered}
-\dot{\bar{\mathbf{P}}} - \dot{\bar{\mathbf{S}}} = \mathbf{Q} + \mathbf{A}^T(\bar{\mathbf{P}} + \bar{\mathbf{S}}) + (\bar{\mathbf{P}} + \bar{\mathbf{S}})\mathbf{A} - \mathbf{M} - \\
\bar{\mathbf{S}}\mathbf{BR}^{-1}\mathbf{B}^T\bar{\mathbf{S}} + \bar{\mathbf{P}}\mathbf{BR}^{-1}\mathbf{B}^T\bar{\mathbf{P}}
\end{gathered} \tag{36}$$

We chose

$$-\dot{\bar{\mathbf{P}}} = -\mathbf{M} + \mathbf{A}^T\bar{\mathbf{P}} + \bar{\mathbf{P}}\mathbf{A} + \bar{\mathbf{P}}\mathbf{BR}^{-1}\mathbf{B}^T\bar{\mathbf{P}} \tag{37}$$

$$\bar{\mathbf{P}}_{t_f} = -\mathbf{G}_f \tag{38}$$

that can be written

$$-(-\dot{\bar{\mathbf{P}}}) = \mathbf{M} + \mathbf{A}^T(-\bar{\mathbf{P}}) + (-\bar{\mathbf{P}})\mathbf{A} - (-\bar{\mathbf{P}})\mathbf{BR}^{-1}\mathbf{B}^T(-\bar{\mathbf{P}}) \tag{39}$$

$$-\bar{\mathbf{P}}_{t_f} = \mathbf{G}_f \tag{40}$$

From (39) and (40), it appears that the function $(-\bar{\mathbf{P}})$ satisfies the Ricatti equation for which the conditions of Theorem 1 represented by

$$\mathbf{M}(t) \geq 0 \quad \forall\, t \in [0, t_f] \tag{41}$$

$$\mathbf{R}^{-1}(t) > 0 \quad \forall\, t \in [0, t_f] \tag{42}$$

$$\mathbf{G}_f \geq 0 \tag{43}$$

are met.

Therefore $(-\bar{\mathbf{P}})$ exists for any $t \in [0, t_f]$.

Replacing the expression (37) in (36) and the boundary constraint (38) in (30) we get

$$-\dot{\bar{\mathbf{S}}} = \mathbf{Q} + \bar{\mathbf{S}}\mathbf{A} + \mathbf{A}^T\bar{\mathbf{S}} - \bar{\mathbf{S}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\bar{\mathbf{S}} \tag{44}$$

respectively

$$\mathbf{\Phi}_f - \bar{\mathbf{P}}(t_f) - \bar{\mathbf{S}}(t_f) = \mathbf{G}_f = -\bar{\mathbf{P}}(t_f) \tag{45}$$

or

$$\bar{\mathbf{S}}(t_f) = \mathbf{\Phi}_f \tag{46}$$

Because (44),(46) are identical to(26),(27) and $\mathbf{P}(t)$ and $\bar{\mathbf{P}}(t)$ exist for any $t \in [0, t_f]$, using (31) it follows that $\bar{\mathbf{S}}(t) = \mathbf{S}(t)$ exists for any $t \in [0, t_f]$.

Thus Theorem 2 is proved. □

The relation between the sufficient conditions for the existence of the solution to the Ricatti equation formulated in the previous theorems is established by

The conditions in Theorem 2 are stronger than those in Theorem 1.

Proof. If replacing $\mathbf{Q}$ and $\mathbf{A}$ with $(\mathbf{Q} - \mathbf{C}^T\mathbf{R}^{-1}\mathbf{C})$ and $(\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{C})$, respectively, the conditions for the existence of the solution for the equations (26),(27) expressed by Theorem 2 come to the construction of a symmetrical matrix $\mathbf{P}(t)$, $t \in [0, t_f]$ so that

$$\mathbf{B}^T\mathbf{P} = 0 \quad \forall t \in [0, t_f] \tag{47}$$

$$\begin{gathered}\dot{\mathbf{P}} + \mathbf{Q} - \mathbf{C}^T\mathbf{R}^{-1}\mathbf{C} + \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{C}) + \\ (\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{C})^T\mathbf{P} = \mathbf{M}(t) \geq 0, \quad \forall t \in [0, t_f]\end{gathered} \tag{48}$$

$$\mathbf{\Phi}_f - \mathbf{P}(t_f) = \mathbf{G}(t_f) \geq 0 \tag{49}$$

If the conditions from Theorem 1 are veri.ed, then the conditions (47),(48),(49) are satis.ed by $\mathbf{P} = 0$, and thus Theorem 3 is proved.

□

5 The optimal solution

Integrating the linear differential equation (17) with the boundary conditions (18) we obtain

$$\mathbf{h}(t) = -\exp\Big(- \int\limits_0^t \mathbf{K}(\tau) d\tau \Big) \cdot \int\limits_t^{t_f} \Big[\exp\Big(\int\limits_0^\tau \mathbf{K}(s) ds \Big) \mathbf{S}(\tau) \mathbf{F}(\tau) \Big] d\tau \tag{50}$$

where

$$\mathbf{K} = -\mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S} + \mathbf{A}^T \tag{51}$$

Cauchy's problem solution for the differential equation (13) with the initial condition $\mathbf{x}_0$ is

$$\mathbf{x}(t) = \exp\Big(\int\limits_0^t \mathbf{X}(\tau) d\tau \Big) \Big[\mathbf{x}_0 + \int\limits_0^t \exp\Big(- \int\limits_0^\tau \mathbf{X}(s) ds \Big) \mathbf{Y}(\tau) d\tau \Big] \tag{52}$$

where we have noted

$$\mathbf{X} = \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{S} \tag{53}$$

$$\mathbf{Y} = -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{h} + \mathbf{F} \tag{54}$$

For the values of $\mathbf{h}(t)$ and $\mathbf{x}(t)$ resulted from(50)and (52) the optimal control becomes

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}^T \big[\mathbf{P}(t)\mathbf{x}(t) + \mathbf{h}(t) \big]. \tag{55}$$

6 Conclusions

The present study proposes a method for the solving of the linear two-point boundary value problem. This is equivalent to the finding of the optimal solution for the static quadratic functionals with differential restrictions represented by the inhomogeneous linear control system. If the adjoint variables are expressed as functions of the state variables, from the necessary extremum conditions, the Ricatti matrix differential equation associated to the optimum problem is obtained. The sufficient conditions for the existence of the solution to the Ricatti equation that ensure a local weak minimum in the analyzed optimal non-singular control are established.

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