Midterm Exam M340

(20p)
① $y'(t) - 2t y(t) = 2t, \quad y(0) = 1$

Homog. sol: $y_H(t) = c e^{t^2}$ $\quad [\, y'(t) - 2ty(t) = 0 \iff \frac{dy}{dt} = 2ty \iff \frac{dy}{y} = 2t\,dt \iff$

$\iff \ln|y| = t^2 + k \iff y = e^{t^2+k} = e^k e^{t^2} = c e^{t^2} \,]$

Particular sol: $y_p(t) = At + B$

$y'(t) - 2ty(t) = 2t$ becomes: $A - 2t(At+B) = 2t \iff -2At^2 - 2Bt + A = 2t$

$\iff \begin{cases} A = 0 \\ B = -1 \end{cases} \quad \Rightarrow y_p(t) = -1$

General sol: $y(t) = c e^{t^2} - 1$; $\quad 1 = y(0) = c e^{0^2} - 1 \Rightarrow c = 2 \Rightarrow \boxed{y(t) = 2e^{t^2} - 1}$

(15p)
② $y''(t) - 3y'(t) + 2y(t) = 0, \quad y(0) = 1, \quad y'(0) = 0$

Characteristic eq:

$r^2 - 3r + 2 = 0 \iff (r-1)(r-2) = 0 \iff r = 1$ or $r = 2$.

General sol:

$y(t) = c_1 e^{1\cdot t} + c_2 e^{2t}$

But $\begin{array}{l} 1 = y(0) = c_1 + c_2 \\ 0 = y'(0) = c_1 + 2c_2 \end{array} \Rightarrow \begin{cases} c_1 = 2 \\ c_2 = -1 \end{cases} \Rightarrow \boxed{y(t) = 2e^t - e^{2t}}$

(15p)
③ $y_1(t) = e^t, \quad y_2(t) = e^t - e^{5t}$

a) $W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^t & e^t - e^{5t} \\ e^t & e^t - 5e^{5t} \end{vmatrix} = e^t(e^t - 5e^{5t}) - e^t(e^t - e^{5t}) =$

$= e^{2t} - 5e^{6t} - e^{2t} + e^{6t} = -4e^{6t} \neq 0 \Rightarrow y_1$ and y_2 are lin. indep.

b) $t\,y''(t) - y(t) = 0$

$y_1(t) = e^t$: $\quad t(e^t)'' - e^t = 0 \iff te^t - e^t = 0 \iff (t-1)e^t = 0$, hence not a sol.

$y_2(t) = e^t - e^{5t}$: $\quad t(e^t - e^{5t})'' - (e^t - e^{5t}) = 0 \iff (t-1)e^t + (1-25t)e^{5t} = 0$, — " — " ——.

(15p)
④ $L[2t+3] = 2L[t] + 3L[1] = 2\frac{1!}{s^{1+1}} + 3\cdot\frac{1}{s} = \frac{2}{s^2} + \frac{3}{s}$

$L[t^2 \underbrace{\sin t}_{f(t)}] = (-1)^2 \cdot \hat{f}''(s) = \left(\frac{1}{s^2+1}\right)'' = \left(-\frac{2s}{(s^2+1)^2}\right)' = -\frac{2(s^2+1)^2 - 2s\cdot 2(s^2+1)\cdot 2s}{(s^2+1)^4} = \frac{2(3s^2-1)}{(s^2+1)^3}$

$L[e^{3t} \underbrace{t^n}_{f(t)}] = \hat{f}(s-3) = \frac{n!}{(s-3)^{n+1}}$

(15p)
⑤ The solution is $y(t) = (B + bA)Y(t) + AY'(t)$, where $A = 1, B = 0, b = -2$,

$Y(t) = L^{-1}\left[\frac{1}{P(s)}\right] = L^{-1}\left[\frac{1}{s^2-2s+1}\right] = L^{-1}\left[\frac{1}{(s-1)^2}\right] = te^t$

Hence, $y(t) = (0 + (-2)1)\cdot te^t + 1\cdot(te^t)' = -2te^t + e^t + te^t = (1-t)e^t$

(20p)
⑥ $y_1(t) = t^4$ is a sol. of $t^2 y''(t) - 2ty'(t) - 4y(t) = 0$. Consider the second sol. of the form $y(t) = t^4 \cdot v(t)$. Plugging into our eq. gives after simplifications: $t^6 v'' + 6t^5 v' = 0$

$\iff t v'' = -6v' \overset{u=v'}{\iff} t u' = -6u \iff \frac{du}{u} = -\frac{6}{t}dt \iff \ln u = -6\ln t$

$\Rightarrow u = \frac{1}{t^6} \Rightarrow v' = \frac{1}{t^6} \Rightarrow v = \frac{1}{t^5}\cdot\left(-\frac{1}{5}\right) \Rightarrow y(t) = t^4 \cdot \frac{1}{t^5}\cdot\left(-\frac{1}{5}\right) \Rightarrow \boxed{y(t) = \frac{1}{t}}$

Easy to see that t^4 and $1/t$ are lin. indep.