Solutions for The Take Home Exam 3

① We have the IVP:

$$y''(t) + \omega^2 \cdot y(t) = f(t), \qquad y(0) = A, \quad y'(0) = B.$$

a) It is clear that the sol. for the homog. eq. is

$$y_H(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

We now use the undetermined coefficient to solve if $f(t) = t + e^{-2t}$.

Assume $y_p(t) = D + Et + Fe^{-2t}$.

Then: $y_p''(t) + \omega^2 y_p(t) = 4Fe^{-2t} + \omega^2(D + Et + Fe^{-2t}) =$

$$= \omega^2 D + \omega^2 E t + (4F + \omega^2 F)e^{-2t} =$$

$$= t + e^{-2t},$$

so

$$\begin{cases} \omega^2 D = 0 \\ \omega^2 E = 1 \\ 4F + \omega^2 F = 1 \end{cases} \Rightarrow \quad D = 0, \quad E = \frac{1}{\omega^2}, \quad F = \frac{1}{4+\omega^2}$$

Therefore, $y_p(t) = \frac{1}{\omega^2} t + \frac{1}{\omega^2+4} e^{-2t}$

The general solution is:

$$y(t) = y_H(t) + y_p(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{1}{\omega^2} t + \frac{1}{\omega^2+4} e^{-2t}$$

Using the initial conditions, we get:

$$\begin{aligned} A &= y(0) = c_1 + \frac{1}{\omega^2+4} \\ B &= y'(0) = c_2 \omega + \frac{1}{\omega^2} - \frac{2}{\omega^2+4} \end{aligned} \quad \Rightarrow \quad \begin{cases} c_1 = A - \frac{1}{\omega^2+4} \\ c_2 = \frac{1}{\omega}\left(\frac{2}{\omega^2+4} - \frac{1}{\omega^2} + B\right) \end{cases}$$

Hence, finally:

$$y(t) = \left(A - \frac{1}{\omega^2+4}\right)\cos(\omega t) + \frac{1}{\omega}\left(B + \frac{2}{\omega^2+4} - \frac{1}{\omega^2}\right)\sin(\omega t) + \frac{1}{\omega^2} t + \frac{1}{\omega^2+4} e^{-2t}$$

b) Using variation of parameter, we get:

$$y_p(t) = c_1(t) \cos(\omega t) + c_2(t) \sin(\omega t)$$

where

$$\begin{cases} c_1'(t) \cos(\omega t) + c_2'(t) \sin(\omega t) = 0 \\ -\omega c_1'(t) \sin(\omega t) + \omega c_2'(t) \cos(\omega t) = f(t) \end{cases},$$

$$f(t) = \begin{cases} -t, & \text{if } 0 < t < 1 \\ t-2, & \text{if } 1 \le t < 2 \\ 0, & \text{if } t \ge 2 \end{cases}$$