M340 - Take Home Exam 1

- SOLUTIONS

1. $y'(t) + 8y(t) = 4 + 50\cos(6t)$, $y(0) = a$ (1)

a) • The homog. solution :

$$y'(t) + 8y(t) = 0$$

$$\frac{dy}{dt} = -8y \Rightarrow \frac{dy}{y} = -8\,dt \Rightarrow \int \frac{1}{y}\,dy = \int(-8)\,dt \Rightarrow$$

$$\Rightarrow \ln|y| = -8t + c_1 \Rightarrow y = c e^{-8t}$$

Hence, $\boxed{y_H(t) = c e^{-8t}}$

• A particular solution: $y_p(t) = c(t)e^{-8t}$ (variation of param.)

(1) becomes: $(c(t)e^{-8t})' + 8\cdot c(t)e^{-8t} = 4 + 50\cos(6t)$

$$c'(t)e^{-8t} + \cancel{c(t)e^{-8t}\cdot(-8)} + \cancel{8c(t)e^{-8t}} = 4 + 50\cos(6t)$$

$$c'(t) = 4e^{+8t} + 50e^{8t}\cos(6t)$$

$$c(t) = 4\underbrace{\int e^{8t}\,dt}_{=\frac{1}{8}e^{8t}} + 50\underbrace{\int e^{8t}\cos(6t)\,dt}_{=\frac{1}{50}e^{8t}(4\cos(6t)+3\sin(6t))}$$

(use integration by parts)

$$\Rightarrow c(t) = e^{8t}\cdot\left(\frac{1}{2} + 4\cos(6t) + 3\sin(6t)\right) \Rightarrow$$

$$\Rightarrow \boxed{y_p(t) = \frac{1}{2} + 4\cos(6t) + 3\sin(6t)}$$

The general solution of (1) is

$$y(t) = \frac{1}{2} + 4\cos(6t) + 3\sin(6t) + c e^{-8t}$$

where $y(0) = a$.

But $y(0) = \frac{1}{2} + 4\cos 0 + 3\sin 0 + c e^0 = \frac{1}{2} + 4 + c = \frac{9}{2} + c$ $\Rightarrow$

$$\Rightarrow c = a - \frac{9}{2}$$

Hence $y(t) = \underbrace{\frac{1}{2} + 4\cos(6t) + 3\sin(6t)}_{\text{the steady state part}} + \underbrace{\left(a - \frac{9}{2}\right)e^{-8t}}_{\text{the transient part}}$