SPLINE WAVELETS ANALYSIS OF RETICULATED FUNCTIONS ON BOUNDED INTERVAL

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This paper deals with the multiresolution analysis of reticulated functions on irregular nets adapted to B-spline scaling functions and wavelets on $\mathbb{L}^2([0,1])$.

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1. INTRODUCTION

The aim of this paper is to present a method of multiresolution analysis for any type of one dimensional reticulated functions, adapted to the cubic spline wavelets on $[0,1]$ of Chui and Quack [1].

It is not possible to perform directly the multiresolution analysis of reticulated functions defined on nets included in $[0,1]$. Then, one must extend the reticulated functions to functions defined on the interval $[0,1]$.

There are many possible extensions. In this paper we introduce an extension operator which transforms a reticulated function into a picewise cubic polynomial continuous function on $[0,1]$. This operator is an interpolant, it has good localization properties and recovers cubic polynomials. Moreover, in some sense, this extension is an essential nonoscillating function.

To perform a multiresolution analysis for the function provided by the extension operator, one has to determinate the scaling coefficients. There are two standard methods; by orthogonal projection on a subspace of scaling functions or by interpolation. Here, we propose a cubic spline interpolation method which preserves the cubic polynomials.

The paper is organized as follows. To the end of this section we briefly review the main properties of cubic spline multiresolution analysis on $\mathbb{L}^2([0,1])$. The next section, Section 2, we introduce the approximation operator of continuous functions in the spaces of B-spline scaling functions. Section 3 is devoted to the extension operator of reticulated functions. Finally, using the operators introduced in Sections 2 and 3, we present two numerical examples of multiresolution analysis for reticulated functions on irregular nets and we comment the results, Section 4.

The cubic spline multiresolution analysis introduced by Chui and Quack [1], consists in a set of closed subspaces $V^j_{[0,1]}$ and $W^j_{[0,1]}$, with $j \in \{j_0, j_0+1, ...\}$ ($j_0 \geq 3$) that exhibit the following properties:

1. $V^j_{[0,1]} \subset V^{j+1}_{[0,1]}$
2. $\overline{\bigcup_{j=3}^{\infty} V^j_{[0,1]}} = \mathbb{L}^2([0,1])$
3. $V^{j+1}_{[0,1]} = V^j_{[0,1]} \bigoplus W^j_{[0,1]}$
4. $V^3_{[0,1]} \bigoplus_{j=3}^{\infty} W^j_{[0,1]} = \mathbb{L}^2([0,1])$

The spaces $V^j_{[0,1]}$ have dimension 2^j+3 and they are generated by the basis $\{\varphi^j_k\}_{-3\leq k\leq 2^j-1}$ of *spline scaling functions.* The spaces $W^j_{[0,1]}$ have dimension 2^j and they are generated by the basis $\{\psi^j_k\}_{-3\leq k\leq 2^j-4}$ of *spline wavelets.*

There are 2^j-3 interior spline scaling functions and 6 boundary spline scaling functions. The basis of $V^j_{[0,1]}$ can be introduced by means of the following functions

$$\begin{aligned} N(x) &:= \frac{1}{6}\sum\nolimits_{l=0}^{4}(-1)^l C_4^l (x-l)_+^3, && (1)\\ B_k(x) &:= \frac{(-1)^{4+k}}{(3+k)!}\sum\nolimits_{l=1}^{4+k}(-1)^l l^k C_{4+k}^l (l-x)_+^3, && (2)\end{aligned}$$

for $k \in \{-3,-2,-1\}$, where

$$x_+ := \begin{cases} x & if \;\; x \geq 0 \\ 0 & if \;\; x < 0. \end{cases}$$

The interior spline scaling functions are given by

$$\varphi^j_k(x) := N(2^j x - k), \quad \text{for} \quad k \in \{0, ..., 2^j-4\}. \tag{3}$$

The boundary scaling spline functions are defined by

$$\begin{aligned} \varphi^j_k(x) &:= B_k(2^j x), && \text{for} \quad k \in \{-3,-2,-1\},\\ \varphi^j_k(x) &:= \varphi^j_{k-2^j}(1-x), && \text{for} \quad k \in \{2^j-3, 2^j-2, 2^j-1\}. \end{aligned} \tag{4}$$

Generally, the explicit formulae of spline wavelets are not very useful. More important are the scaling relations between two levels of resolution

$$\begin{aligned} \varphi^j_k(x) &= \sum_{l=-3}^{2^{j+1}-1} p_{k,l}\varphi^{j+1}_l(x), \quad \text{for} \quad k=-3, 2^j-1,\\ \psi^j_k(x) &:= \sum_{l=-3}^{2^{j+1}-1} q_{k,l}\varphi^{j+1}_l(x), \quad \text{for} \quad k=-3, 2^j-4, \end{aligned} \tag{5}$$

with $p_{k,l}$ and $q_{k,l}$ simple band matrices (for details, see [1, 7]).

Any function $f \in V_{[0,1]}^{j+1}$ can be written

$$f(x) = \sum_{l=-3}^{2^j-1} f_l^j \varphi_l^j(x) + \sum_{l=-3}^{2^j-4} d_l^j \psi_l^j(x), \tag{6}$$

where f_l^j and d_l^j are the scaling coefficients and the wavelet coefficients, respectively. The reconstruction algorithm from level j to level $j+1$ is given by the following relation

$$f_l^{j+1} = \sum_{k=-3}^{2^j-1} f_k^j p_{k,l} + \sum_{k=-3}^{2^j-4} d_k^j q_{k,l}. \tag{7}$$

We mention that, since the spline scaling functions and the spline wavelets are not orthogonal bases, the deconstruction algorithm uses dual bases.

2. $V_{[0,1]}^j$ APPROXIMATION OF CONTINUOUS FUNCTIONS

Our aim is to construct an approximation operator which interpolates the values of the functions in points of the form $k/2^j$ and whose restriction to the subspace of cubic polynomials is the identity.

In this frame, the last request ensures a maximal degree of accuracy, similar to [4], for orthogonal projections on the scaling functions subspaces.

Let $M = 2^j + 1$ and let $C[0,1]$ be the space of real continuous functions on $[0,1]$. We look for $P_j : C[0,1] \to V_{[0,1]}^j$ such that:

p$_1$) $(P_j f)\left(\dfrac{k}{2^j}\right) = f\left(\dfrac{k}{2^j}\right)$ for all $k \in \{0,1,2,...,M-1\}$.

p$_2$) $P_j f = f$ for all $f \in \pi_3$,

with π_3, the space of cubic polynomials restricted to $[0,1]$.

It is more convenient to work in the alternative basis $\{N(2^j \cdot - k)\}_{-3 \le k \le M-2}$. Denote by b_l the coefficients of $P_j f$ in this basis. From (p$_1$), we obtain a linear system of M equations with $M+2$ unknowns b_l.

Let $y_k := f((k-1)/2^j)$ and $A_{k,l} := 6N(k+2-l)$. We consider the following system of M equations and M unknowns b_l, with $-2 \le l \le M-3$

$$\sum_{l=1}^{M} A_{k,l} b_{l-3} = 6y_k - 6b_{-3}N(k+2) - 6b_{M-2}N(k-2^j) \tag{8}$$

where $k \in \{1,...,M\}$.

Let A be the matrix of elements $A_{k,l}$, for $1 \le k,l \le M$. The inverse of A has the form

$$(A^{-1})_{k,l} = \begin{cases} \frac{\lambda^{l-k+1}(\lambda^{2k}-1)(1-\lambda^{2(M+1-l)})}{(1-\lambda^2)(1-\lambda^{2(M+1)})} & \text{for } k < l \\ \\ \frac{\lambda^{k-l+1}(\lambda^{2l}-1)(1-\lambda^{2(M+1-k)})}{(1-\lambda^2)(1-\lambda^{2(M+1)})} & \text{for } k \geq l \end{cases}, \tag{9}$$

where $\lambda = -2 + \sqrt{3}$. Note that the elements of A^{-1} have exponential decay from the main diagonal.

Using A^{-1} we define an operator P_j which satisfies property (p_1)

$$(P_j f)(x) = \alpha\zeta(x) + \beta\xi(x) + \sum_{l=1}^{M} y_l \varepsilon_l(x). \tag{10}$$

The functions ζ, ξ, ε_l in (10) are given by

$$\begin{aligned} \zeta(x) &:= N(2^j x + 3) - \sum_{k=1}^{M} (A^{-1})_{k,1} N(2^j x - k + 3), \\ \xi(x) &:= N(2^j x - 2^j + 1) - \sum_{k=1}^{M} (A^{-1})_{k,M} N(2^j x - k + 3), \\ \varepsilon_l(x) &:= 6 \sum_{k=1}^{M} (A^{-1})_{k,l} N(2^j x - k + 3) \quad \text{for} \quad l \in \{1, ..., M\}. \end{aligned} \tag{11}$$

Remark that ζ and ξ vanish in knots and ε_l is interpolant (with ε_l the analogous of fundamental spline of order 4 introduced by Schöenberg [8]). The unknown factors α and β can be determined from condition (p_2) as it follows. In order to calculate the second order derivative of $P_j f$, we introduce the matrix H of elements

$$H_{l,k} := N^{''}(l - k + 2). \tag{12}$$

As $H = A - 6I$, one can easily verify that H commutes with A^{-1}. Using the derivation formula for the functions N, Rel.(10) and the commutation property between H and A^{-1}, for any $k \in \{1, ..., M\}$, we find

$$2^{-2j}(P_j f)^{''}\left(\frac{k-1}{2^j}\right) = 6 \sum_{l=1}^{M} A^{-1}_{k,l} \left(\alpha\delta_{l,1} + \beta\delta_{l,M} + \sum_{n=1}^{M} H_{l,n} y_n\right). \tag{13}$$

In order to satisfy condition (p_2) we look for α and β as linear combinations of $\{y_l\}_{1 \leq l \leq M}$ such that

$$2^{-2j} f^{''}\left(\frac{k-1}{2^j}\right) \equiv 6 \sum_{l=1}^{M} A^{-1}_{k,l} \left(\alpha\delta_{l,1} + \beta\delta_{l,M} + \sum_{n=1}^{M} H_{l,n} y_n\right). \tag{14}$$

for any $f \in \pi_3$. Choosing

$$\begin{aligned} \alpha(y) &:= (21y_1 - 28y_2 + 17y_3 - 4y_4)/6, \\ \beta(y) &:= (21y_M - 28y_{M-1} + 17y_{M-3} - 4y_{M-4})/6 \end{aligned} \tag{15}$$

one can verify that (14) holds.

Using the equality $6A^{-1} = I - 1/6H + 1/6A^{-1}H^2$ and returning to the scaling functions φ_k^j one obtains

$$P_j f = B_j f + Q_j f, \tag{16}$$

where the scaling coefficients of $B_j f$ are given by

$$\begin{array}{lcl} (B_j f)_{-3} & = & y_1, \\ (B_j f)_{-2} & = & (7y_1 + 18y_2 - 9y_3 + 2y_4)/18, \\ (B_j f)_{k-3} & = & (-y_{k-1} + 8y_k - y_{k+1})/6 \quad \text{for} \quad k \in \{2, ..., M-1\}, \\ (B_j f)_{M-3} & = & (7y_M + 18y_{M-1} - 9y_{M-2} + 2y_{M-3})/18, \\ (B_j f)_{M-2} & = & y_M, \end{array} \tag{17}$$

and the scaling coefficients of $Q_j f$ have the form

$$\begin{array}{lcl} (Q_j f)_{-3} & = & (Qf)_{M-2} = 0, \\ (Q_j f)_{-2} & = & -\sum_{l=3}^{M-2} (A^{-1})_{1,l} \Delta_l(f)/9, \\ (Qf)_{2^j-2} & = & -\sum_{l=3}^{M-2} (A^{-1})_{M,l} \Delta_l(f)/9, \\ (Q_j f)_{k-3} & = & \sum_{l=3}^{M-2} (A^{-1})_{k,l} \Delta_l(f)/6 \quad \text{for} \quad k \in \{2, ..., M-1\}, \end{array} \tag{18}$$

with $\Delta_l(f) := y_{l-2} - 4y_{l-1} + 6y_l - 4y_{l+1} + y_{l+2}$.

Since $B_j(f) = f$ and $Q_j(f) = 0$ for each $f \in \pi_3$, obviously P_j satisfies (p_2).

Denote by $\omega(f, \eta)$ the continuity module of f

$$\omega(f, \eta) := \sup_{|x-y| \leq \eta} |f(x) - f(y)| \tag{19}$$

and let $C^r[0,1]$ be the space of C^r real functions on $[0,1]$.

The properties of P_j (as operator from $C[0,1]$ into $C[0,1]$) are summarized by the following theorem.

THEOREM 1. *For each* $f \in C[0,1]$

i) $\|P_j\|_\infty \leq 3$,

ii) $\|f - P_j f\|_\infty \leq 3\omega(f, 2^{-j})$.

If $f \in C^{r+1}[0,1]$, *with* $r \in \{0,1,2,3\}$

iii) $\|f - P_j f\|_\infty \leq c \cdot 2^{-j(r+1)} \left\| f^{(r+1)} \right\|_\infty$,

Proof. It is known that φ_l^j is partition of unity [1]. It is easily checked that

$$\sum_{l=1}^{M} \left| A_{k,l}^{-1} \right| \leq \frac{1}{2}. \tag{20}$$

i) Let $f_l^j := (B_j f)_l + (Q_j f)_l$ (for $-3 \leq l \leq M-2$). Therefore, we have

$$|P_j f(x)| \leq \sum_{l=-3}^{M-2} \left| f_l^j \right| \varphi_l^j(x) \leq \max_{-3 \leq l \leq M-2} \left| f_l^j \right|. \tag{21}$$

Using (17), (18) and (20) one can write

$$\left| f_l^j \right| \leq 3 \left\| f \right\|_\infty. \tag{22}$$

Combining (21) with (22) we conclude (i).

ii) Involving simple estimations, again by (17), (18) and (20) one obtains

$$\begin{aligned}
&\Delta_l(f) \leq 8\omega(f, 2^{-j}), && \text{for} \quad l \in \{3, ..., M-2\}, \\
&\left| f_{-2}^j - y_2 \right| \leq \omega(f, 2^{-j}), && \\
&\left| f_{k-3}^j - y_k \right| \leq \omega(f, 2^{-j}), && \text{for} \quad k \in \{2, ..., M-1\}, \\
&\left| f_{M-3}^j - y_{M-1} \right| \leq \omega(f, 2^{-j}). &&
\end{aligned} \tag{23}$$

Then, for a fixed k and $x \in ((k-1)/2^j, k/2^j)$ we have

$$\begin{aligned}
&|f(x) - P_j(f)(x)| \leq \sum_{l=k-4}^{k-1} \left| f(x) - f_l^j \varphi_l^j(x) \right| \\
&\leq \max_{k-4 \leq l \leq k-1} \left| f(x) - f_l^j \right| \leq 3\omega(f, 2^{-j}).
\end{aligned} \tag{24}$$

iii) Since P_j is a bounded linear operator, from Taylor's formula and from (p_2) we can write Peano's formula

$$(I - P_j) f(x) = \frac{1}{r!} \int_0^1 [(I - P_j)(\cdot - y)_+^r](x) f^{(r+1)}(y) dy. \tag{25}$$

Let

$$\begin{aligned}
K_1(x, y) &:= [(I - B_j)(\cdot - y)_+^r](x), \\
K_2(x, y) &:= [-Q_j(\cdot - y)_+^r](x).
\end{aligned} \tag{26}$$

Then we have

$$|(I - P_j) f(x)| \leq \frac{1}{r!} \left\| f^{(r+1)} \right\|_\infty \left(\int_0^1 |K_1(x, y)| \, dy + \int_0^1 |K_2(x, y)| \, dy \right). \tag{27}$$

For the second integral in the r.h.s. of Rel.(27), using that *supp* $\Delta_l(\cdot - y)_+^r = [(l-3)/2^j, (l+1)/2^j]$, one obtains

$$\int_0^1 |K_2(x, y)| \, dy \leq \sum_{l=3}^{M-2} \left| A_{k,l}^{-1} \right| \int_{\frac{l-3}{2^j}}^{\frac{l+1}{2^j}} \left| \Delta_l(\cdot - y)_+^r \right| dy \leq c \cdot 2^{-j(r+1)}.$$

The first integral in the r.h.s. of Rel.(27) can be estimated as follows. Consider $x \in ((k-1)/2^j, k/2^j)$. Using properties of the scaling functions supports, we can write

$$B_j((\cdot - y)_+^r)(x) = \sum_{l=k-4}^{k-1} B_j((\cdot - y)_+^r)_l \varphi_l^j(x).$$

Observe that the scaling coefficients of $B_j((\cdot - y)_+^r)_l$, for $k-4 \leq l \leq k-1$, coincide to the scaling coefficients of $(\cdot - y)^r$ whenever $y \leq (k-3)/2^j$. Moreover, these coefficients vanish for $y \geq (k+2)/2^j$. Consequently, we have

$$\int_0^1 |K_1(x,y)|\, dy = \int_{\frac{k-3}{2^j}}^{\frac{k+2}{2^j}} \left| (x-y)_+^r - \sum_{l=k-4}^{k-1} B_j((\cdot - y)_+^r)_l \cdot \varphi_l^j(x) \right| dy$$
$$\leq \sum_{l=k-4}^{k-1} \left[\int_{\frac{k-3}{2^j}}^{\frac{k+2}{2^j}} \left| (x-y)_+^r - B_j((\cdot - y)_+^r)_l \right| dy \right] \cdot \varphi_l^j(x) \leq c \cdot 2^{-j(r+1)},$$

for $k \in \{3, ..., 2^j - 2\}$. For the reminder values of k, the estimation follows similarly. □

Denote by J the maximal level of resolution. Let $j_0 \leq J$. Consider the following representation of $P_J f$

$$P_J f = \sum_{l=-3}^{2^{j_0}-1} \tilde{f}_l^{j_0} \varphi_l^{j_0} + \sum_{j=j_0}^{J-1} \sum_{l=-3}^{2^j-4} \tilde{d}_l^j \psi_l^j, \tag{28}$$

where $\tilde{f}_l^{j_0}$ are the scaling coefficients of $P_J f$ on the level j_0 and $\tilde{d}_l^j$ are the wavelet coefficients of $P_J f$ on the level j obtained by the deconstruction algorithm [7] and which satisfy the following equality.

$$\langle P_J f, \psi_k^j \rangle = \sum_{l=-3}^{2^j-4} \tilde{d}_l^j \langle \psi_l^j, \psi_k^j \rangle. \tag{29}$$

Notice that the values of the wavelet coefficients $\tilde{d}_l^j$ depend also on the maximal level of resoluton J. The magnitude of the wavelet coefficients is controlled by following estimation.

THEOREM 2. *Let $f \in C^{r+1}[0,1]$, with $r \in \{0,1,2,3\}$. Then there exists a positive constant c such that*

$$\tilde{d}_l^j \leq c \cdot (2^{-j(r+1)} + 2^{-J(r+1)}) \left\| f^{(r+1)} \right\|_\infty, \tag{30}$$

for all $j \in \{j_0, ..., J-1\}$.

Proof. We have

$$\left| \left\langle P_J f, \psi_l^j \right\rangle \right| \leq \|f - P_J f\|_\infty \int_0^1 \left| \psi_l^j(x) \right| dx + \left| \left\langle f, \psi_l^j \right\rangle \right|, \tag{31}$$

Remark that wavelet coefficients of polynomials of third degree are zero. Then, by Taylor's formula, for the second term in r.h.s. of (31) we can write

$$\begin{aligned} \left|\left\langle f,\psi_l^j\right\rangle\right| &\leq \frac{1}{r!}\int_0^1\left[\int_0^1 (x-y)_+^r\left|f^{(r+1)}(y)\right|dy\right]\left|\psi_l^j(x)\right|dx \\ &\leq c\cdot\left\|f^{(r+1)}\right\|_\infty 2^{-j(r+2)}, \end{aligned} \tag{32}$$

where we have used the known wavelets property [1]

$$supp\ \psi_l^j \leq 7/2^j. \tag{33}$$

From (31), (32) and point (**iii**) in Theorem 1, for all $j\in\{j_0,...,J-1\}$ one obtains

$$\left|\left\langle P_J f,\psi_l^j\right\rangle\right| \leq c\cdot 2^{-j}\left(2^{-J(r+1)}+2^{-j(r+1)}\right)\left\|f^{(r+1)}\right\|_\infty. \tag{34}$$

Let w^j be the matrix of elements $\left\langle\psi_l^j,\psi_k^j\right\rangle$. Since w^j is diagonal dominant, one can find a positive constant c such that

$$\left\|[w^j]^{-1}\right\|_\infty < c\cdot 2^j. \tag{35}$$

The above estimation can be obtained using, by example, Gerschgorin's Circle Theorem [9], or Theorem 1.2 in [5].

From (29), (34) and (35) we conclude (30). □

Since $\psi_l^j\in V_{[0,1]}^{j+1}$, then wavelets can be expanded in the basis of the scaling functions of $V_{[0,1]}^{j+1}$ as in (5). Let $c_l^j := \sum_{k=-3}^{2^j-4}\tilde{d}_k^j q_{k,l}$. From (28), we find the following useful representation of $P_J f$.

$$P_J f = \sum_{l=-3}^{2^{j_0}-1}\tilde{f}_l^{j_0}\varphi_l^{j_0} + \sum_{j=j_0}^{J-1}\sum_{l=-3}^{2^{j+1}-1} c_l^j\varphi_l^{j+1}. \tag{36}$$

3. $V_{[0,1]}^j-$ APPROXIMATION OF RETICULATED FUNCTIONS

Let $\mathcal{X} := \{x_l\}_{1\leq l\leq n}$ be a net in the interval $[0,1]$, (i.e. $0 = x_1 < x_2 < ... < x_{n-1} < x_n = 1$). Denote by $\mathcal{R} := \{g:\mathcal{X}\to\mathbb{R}\}$ the space of reticulated functions. It is known that the Lagrange interpolation problem can be solved if the net $\mathcal{X}$ satisfies Schönberg-Whitney conditions [6], with respect to spline scaling functions in $V_{[0,1]}^j$.

Our purposes are to avoid any restriction on the knots and to extend the reticulated function to an element of $V_{[0,1]}^j$.

First we extend the reticulated function to an continuous, picewise polynomial function and then, we apply the projection operator P_j defined in the previous section, to obtain an element in $V^j_{[0,1]}$.

We consider the restriction operator $R : C[0,1] \to \mathcal{R}$ with $R(g)(x_i) = g(x_i)$ (for all $1 \le i \le n$). Any application $L : \mathcal{R} \to C[0,1]$ is called extension operator. We say that L has accuracy of order r, if it fulfills there exists a positive constant c, such that

$$\|(L \circ R)(g) - g\|_\infty \le ch^r \left\| g^{(r)} \right\|_\infty \quad (\forall)\ g \in C^r[0,1], \tag{37}$$

where $h = h(\mathcal{X}) := \max_{1 \le l \le n-1} |x_{l+1} - x_l|$.

The control on errors of approximation are given by the following proposition:

Proposition 1.

i) *For each $g \in C^r[0,1]$ and L with accuracy of order r, $(1 \le r \le 3)$, there exists a positive constant c such that:*

$$\|(P_j \circ L \circ R)(g) - g\|_\infty \le c \cdot (2^{-jr} + h^r) \left\| g^{(r)} \right\|_\infty . \tag{38}$$

ii) *For any $g \in \mathcal{R}$*

$$|(P_j \circ L)(g)(x_i) - g(x_i)| \le c \cdot \omega((L(g), 2^{-j}). \tag{39}$$

The proof follows immediately from Theorem 1 and from definition (37).

It is known that Gibbs phenomenon can be present in the modeling of discontinuous processes by functions with certain regularity properties. There are different criteria for minimizing Gibbs oscillations (see [2, 3], by example). Since Gibbs phenomenon does not appear for linear interpolation, it is natural to control the deviation of the extended function from linear interpolation. We introduce the following measure of oscillations for the extension operator L

$$\mu_g(L) := \|L(g) - L_0(g)\|_{\mathbb{L}^2} , \tag{40}$$

with L_0, the operator of linear spline interpolation.

We look for a cubic spline extension operator with the following properties:

l_1) $L(g)(x_i) = g(x_i)$, for any $g \in \mathcal{R}$.
l_2) $(L \circ R)(g) = g$, for all $g \in \pi_3$.
l_3) L is a local operator.
l_4) L minimizes μ_g given in (40).

For any interval $[x_k, x_{k+1}]$, consider the consecutive knots x_k, x_{k+1}, x_l, x_m (not necessary ordered). We start from the Newton's formula of Lagrange interpolation polynomial

$$\begin{aligned} K^k_{l,m}(g)(x) := g(x_k) \quad & +(x - x_k)[x_k; x_{k+1}]g \\ & +(x - x_k)(x - x_{k+1})[x_k; x_{k+1}; x_l]g \\ & +(x - x_k)(x - x_{k+1})(x - x_l)[x_k; x_{k+1}; x_l; x_m]g. \end{aligned} \tag{41}$$

Here $[\cdot;\cdot]$, $[\cdot;\cdot;\cdot]$ and $[\cdot;\cdot;\cdot;\cdot]$ represent the divided differences operators of order one, two and three, respectively. For each interval $[x_k, x_{k+1}]$ (except boundaries) there exist three polynomials defined by (41). We choose that pair l and m which minimizes the following norm:

$$d^k_{l,m} := \left\| K^k_{l,m}(g) - L_0(g) \right\|_{\mathbb{L}^2[x_k, x_{k+1}]} . \tag{42}$$

Standard computations show that the expression of $d^k_{l,m}$ has the following simple form

$$d^k_{l,m} = \{(x_{k+1} - x_k)^5[(\lambda^k_{l,m})^2 + (\lambda^{k+1}_{l,m})^2 + 3/2 \cdot \lambda^k_{l,m}\lambda^{k+1}_{l,m}]/105\}^{1/2}, \tag{43}$$

where

$$\begin{aligned} \lambda^k_{l,m} &:= (x_k - x_l)[x_k; x_{k+1}; x_l; x_m]g + [x_k; x_{k+1}; x_l]g, \\ \lambda^{k+1}_{l,m} &:= (x_{k+1} - x_l)[x_k; x_{k+1}; x_l; x_m]g + [x_k; x_{k+1}; x_l]g. \end{aligned} \tag{44}$$

From (41) and (44) we can also write

$$\begin{aligned} &K^k_{l,m}(g)(x) = g(x_k) + (x - x_k)([x_k; x_{k+1}]g - (x_{k+1} - x_k)\lambda^k_{l,m}) \\ &+(x - x_k)^2(2\lambda^k_{l,m} - \lambda^{k+1}_{l,m}) + (x - x_k)^3(\lambda^{k+1}_{l,m} - \lambda^k_{l,m})/(x_{k+1} - x_k). \end{aligned} \tag{45}$$

For any $x \in [x_k, x_{k+1}]$ we define the extension operator L such that

$$L(g)(x) := K^k_{l,m}(g)(x) \tag{46}$$

with l and m such that, $d^k_{l,m}$ in (43) takes the minimum value. It is easily checked that L verifies properties (l_1-l_4).

4. NUMERICAL APPLICATIONS

In this section we present two simple examples of multiresolution analysis for reticulated functions on irregular nets. In each example, using the results presented in Sections 2 and 3 one proceeds as it follows:

1) One transforms a reticulated function into a continuous function by means of the extension operator L, defined by Rel.(46).
2) This continuous function is approximated with an element of $V^J_{[0,1]}$ using the operator P_J given by (16).
3) One performs the multiresolution analysis.

In the following examples, $J = 15$ is the highest level of resolution for the multiresolution analysis.

Example 1. We consider a reticulated function f defined as it follows:

x	0.0000	0.0796	0.0989	0.2288	0.2587	0.3266	0.4343
f(x)	-1	-1	-1	-1	-1	-1	1
x	0.4543	0.4702	0.4826	0.6176	0.6718	0.7035	0.7199
f(x)	1	1	1	-1	1	1	1
x	0.7314	0.7414	0.7432	0.8963	0.9143	0.9574	1.0000
f(x)	1	1	-1	-1	-1	-1	-1

In this example $\mu_f(L) = 9.61 \cdot 10^{-2}$, while for the classical spline interpolation μ_f =7.28, with μ is given by (40). Obviously, $\mu_f(P_J \circ L) \stackrel{J\to\infty}{\longrightarrow} \mu_f(L)$. This property is also verified numerically as one can see in the following table.

J	6	8	12	15	20
$\lvert\mu_f(P_J \circ L) - \mu_f(L)\rvert$	1.95	$2.92 \cdot 10^{-3}$	$3.01 \cdot 10^{-5}$	$1.58 \cdot 10^{-7}$	10^{-11}

Figure 1 presents the graphics of f, the extension of f on the level 8 of resolution and a classical spline interpolation. Figure 2 presents a zoom in graphics of f and the extension of f on the levels of resolution 4, 6 and 8.

Example 2. Mimicking $\sin(1/x)$ with the function

$$g(x) := \sin[100\pi/(50x + 1)] \tag{47}$$

we use g to generate a reticulated function $y_k := g(x_k)$ on the net $x_k := 10/(501 - k) - 0.2$, for $k \in \{1, ..., 491\}$ and $x_{492} := 1$.

In Figures 3 and 4 we compare the extension of g on the levels 12 and 8 respectively, with the extension of g on the level 15.

Notice that for a sufficiently large level of resolution, the extended function does not introduce significantly additional oscillations (as classical spline interpolation) Example 1, whereas if the reticulated function becomes from an oscillating function, Example 2, the extended function is an oscillating function too.

As in the case of dyadic nets, the multiresolution analysis of reticulated functions on irregular nets can provide an economic storage of information by data compression. A compression can be achieved by removing those terms in (36) for which $|c_l^j| < \varepsilon$, where $\varepsilon > 0$ is a given tolerance.

For exemplification, we consider again the function g defined by (47), restricted to an equidistant net $x_k := (k - 1)/n$ of n points.

Denote by $\tilde{g}$ the approximation of $(P_J \circ L)(g)(x_k)$ for a tolerance ε and let τ be the number of "significant" terms corresponding to this tolerance.

Let $\delta := \max\{|\tilde{g}(x_k) - g(x_k)|, 1 \leq k \leq n\}$ and let $\gamma := (1 - \tau/n) \times 100$ be the compression factor. For $n = 10001$, we present some compression results in the following table.

1	$J = 15$		$J = 18$		$J = 20$	
ε	δ	γ	δ	γ	δ	γ
10^{-1}	0.134	96.5	0.134	96.5	0.134	96.0
10^{-2}	0.035	92.5	0.020	92	0.020	92
10^{-3}	0.034	85.7	0.005	78	0.002	78

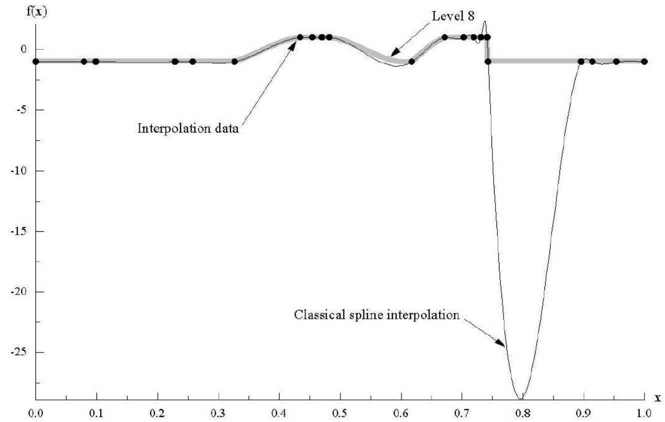

Figure 1. The function f, the extension of f on the level 8 of resolution and the classical spline interpolation.

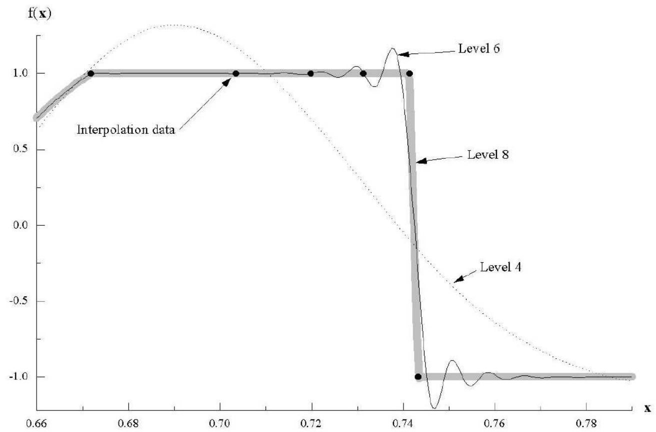

Figure 2. The function f and the extension of f on the levels of resolution 4, 6 and 8.

Final Remarks. The function provided by the extension operator L is only of C^0 class. By projection on $V^j_{[0,1]}$, using the operator P_j, one obtains a C^2 class function with the accurancy controlled by the estimation (38) in Proposition 1.

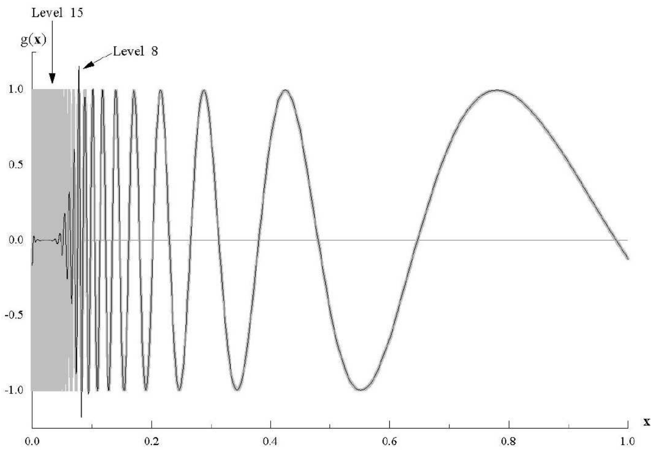

Figure 3. The extension of g on the levels of resolution 8 and 15.

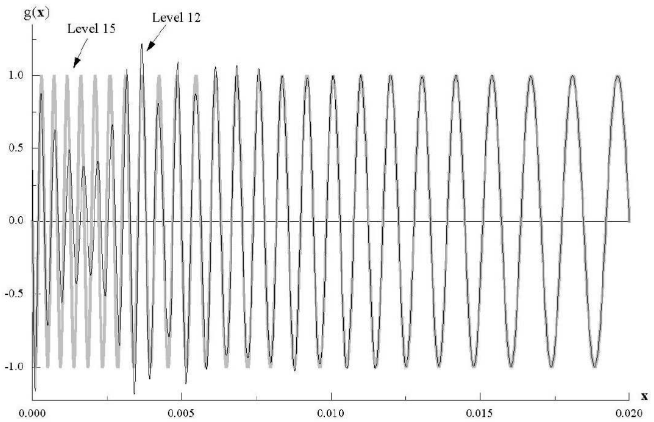

Figure 4. The extension of g on the levels of resolution 12 and 15.

Notice that, although L and P_j are interpolants, the operator $P_j \circ L$ is not. However, the values of the reticulated function can be recovered with any accurancy for a level of resolution j big enough. Since $(L(g))$ is picewise derivable, from Rel. (39) we have

$$|(P_j \circ L)(g)(x_i) - g(x_i)| \leq c \cdot 2^{-j} \left\| (L(g))' \right\|_\infty,$$

with $(L(g))'$ defined on its natural domain.

In (40) one can choose other criteria to measure the deviation from linear spline interpolation (by example, $\mathbb{L}^p$-norm or the total variation), but the $\mathbb{L}^2$-norm seems to be more natural for the multiresolution analysis of $\mathbb{L}^2([0,1])$ and in the class of cubic spline extensions with properties (l_1-l_3) due to condition (l_4), the operator L can be considered, in $\mathbb{L}^2$ sense, an essential nonoscillating extension.

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