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Fast solvers for EVSS formulations of viscoelastic flows

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Numerical simulations of viscoelastic flow problems require the solution of large, typically sparse, systems of equations which inherit the (highly) nonlinear coupling of the original PDE model. Recent approaches, such as the elastic viscous split stress (EVSS) methods, while allowing more flexibility and the treatment of complex rheological models come at the cost of increased problem size. At the same time, there is an evident lack of efficient solution techniques. In this work we introduce and analyze a class of implicit iterative solvers based on a Schur complement approach. The technique is based on identifying a suitable decoupling of the original system of PDE through an approximation to a Schur complement operator. We illustrate our approach on discretizations of EVSS formulations of Oldroyd-type fluid models.

1. Introduction

The numerical treatment of viscoelastic flow models has seen in recent years an increase in the research effort. This is largely due to the seminal paper of Rajagopalan et. al [16] which proposed the introduction of the rate of deformation tensor as an unknown, leading to a rather versatile class of elastic-viscous split stress or EVSS methods. Thereafter, a series of papers ensued which outlined analysis of the method as well as numerical experiments [8, 7, 2, 6, 3]. We note here that the favoured numerical approach is mixed finite element methods, due both to their flexibility and amenability to analysis.
The increased versatility of the EVSS approach comes, however, at a computational cost. The problems are larger and they depart from the classical Stokes and Navier-Stokes setup for which a great number of optimal and quasi-optimal, linear and nonlinear system solvers have been devised over the years. While some work has been

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carried out in this respect [10], this issue remains to be investigated, especially given the complexity of the numerical simulations: in three-dimensions, for example, we require the solution of a nonlinear coupled system of PDE in ten variables.

Our aim in this paper is to employ previous results on preconditioning for systems of PDE [4, 12] to derive a useful iterative method for EVSS formulations. This approach is essentially a Schur complement approach which incorporates information derived from consideration of the symbol of the operator. The paper is organized as follows. In section 2 we describe the EVSS method and review briefly an associated mixed finite element discretization. Section 3 outlines a general Schur complement approach for elliptic systems of PDE and derives for the EVSS operator the velocity-pressure Schur complement which is our essential ingredient in the preconditioning technique we propose. Finally, in section 4, we discuss briefly the implementation details.

2. Problem formulation

Let Ω denote an open subset of $\mathbb{R}^n$ with Lipschitz boundary Γ and consider the following generic steady-state non-Newtonian model

$$\left\{\begin{array}{ll} -\operatorname{div}\sigma + u\cdot\nabla u & = f \\ \operatorname{div} u & = 0 \end{array}\right. \qquad \text{in } \Omega \tag{1}$$

together with boundary conditions, where the stress tensor σ is given by

$$\sigma = -pI_n + \sigma_e,$$

with σ_e the extra stress tensor. In the case of a Stokes model for a polymeric solute, the extra stress tensor is the sum of viscous and elastic contributions

$$\sigma_e = \sigma_V + \sigma_E,$$

where

$$\sigma_E = 2\eta_1 \operatorname{div}\epsilon(u), \quad \sigma_V = 2\eta_2 \operatorname{div}\epsilon(u)$$

with $\epsilon(u) = (\nabla u + \nabla u^T)/2$ and η_1, η_2 the polymer and solvent viscosities, respectively. For an Oldroyd model with a single relaxation time, the elastic contribution is defined implicitly via the constitutive equation

$$\sigma_E = 2\eta_1 \operatorname{div}\epsilon(u) + \lambda B(u, \sigma_E),$$

where

$$B(u,\sigma) = u\cdot\nabla\sigma - \nabla u\sigma - \sigma\nabla u^{\mathrm{T}}.$$

The elastic-viscous stress splitting method (or EVSS) introduces σ_E as an additional variable. The resulting problem reads

$$\left\{\begin{array}{lll} \sigma_E - 2\eta_1\epsilon(u) + \lambda B(u,\sigma_E) & = 0, & \\ -2\eta_2 \operatorname{div}\epsilon(u) + u\cdot\nabla u + \nabla p - \nabla\cdot\sigma_E & = f & \text{in } \Omega \\ \operatorname{div} u & = 0 & \\ u & = u^* & \text{on } \Gamma \\ \sigma & = \sigma^* & \text{on } \Gamma_{in}, \end{array}\right. \tag{2}$$

where for simplicity we chose to work with Dirichlet boundary conditions. Here, $\Gamma_{in} = \{x \in \Gamma : n \cdot u^* < 0\}$ is the inflow boundary. Linearizing around some known value b of u and using the incompressibility condition we arrive at the following problem (we also drop the subscript of σ_E) in variables σ, u, p

$$\begin{cases} \sigma - 2\eta_1\epsilon(u) + \lambda B(b,\sigma) & = 0, & \\ -\eta_2\Delta u + b\cdot\nabla u + \nabla p - \nabla\cdot\sigma & = f & \text{in } \Omega, \\ \operatorname{div} u & = 0 & \\ u & = u^* & \text{on } \Gamma, \\ \sigma & = \sigma^* & \text{on } \Gamma_{in}. \end{cases} \tag{3}$$

We note here that the number of variables in $\mathbb{R}^n$ is $\frac{n(n+1)}{2} + n + 1$.

2.1. Mixed finite element discretizations

Let $H^m(\Omega)$ denote the Sobolev spaces of index 2 with $H^0(\Omega) = L^2(\Omega)$ and let

$$S = [H^1(\Omega)]^{n\times n}_{sym} = \{\sigma_{ij} \in H^1(\Omega) : \sigma_{ij} = \sigma_{ji},\ i,j \le n\},$$
$$V = [H^1_0(\Omega)]^n = \{v_i \in H^1(\Omega) : v_i \mid_\Gamma = 0, i = 1,\dots,n\},$$
$$Q = L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega p(x)\mathrm{d}x = 0\}$$

A mixed variational formulation for this problem reads

$$\begin{cases} \text{Find } (\sigma,u,p) \in X := S\times V\times Q \text{ such that for all } (\tau,v,q)\in X \\ \qquad d(\sigma,\tau) + c(u,\tau) = 0 \\ \qquad c(v,\sigma) + a(u,v) + b(v,p) = f(v) \\ \qquad b(u,q) = 0, \end{cases} \tag{4}$$

where

$$d(\sigma,\tau) = -\frac{1}{2\eta_1}\int_\Omega (\sigma + \lambda B(b,\sigma)) : \tau \mathrm{d}x, \quad c(u,\tau) = \int_\Omega \epsilon(u) : \tau \mathrm{d}x,$$

$$a(u,v) = \int_\Omega (2\eta_2\epsilon(u) : \epsilon(v) + b\cdot\nabla u\ v)\mathrm{d}x, \quad b(v,p) = \int_\Omega v\cdot\nabla p \mathrm{d}x, \quad f(v) = \int_\Omega f\ v\mathrm{d}x.$$

Given a subdivision $\Omega_h = \cup_{k=1}^N T_k$ of Ω into simplices T_k, let

$$\begin{aligned} S_h &= \{\sigma \in S : \sigma \in [P_1(T)]^{n\times n}, T \in \Omega_h\}, \\ V_h &= \{v \in V : v \in [P_2(T)]^n, T \in \Omega_h\}, \\ Q_h &= \{q \in Q : q \in P_1(T), T \in \Omega_h\}. \end{aligned}$$

Writing $X_h = S_h \times V_h \times Q_h$, the corresponding finite element formulation reads

$$\begin{cases} \text{Find } (\sigma_h,u_h,p_h) \in X_h \text{ such that for all } (\tau_h,v_h,q_h)\in X_h \\ \qquad d(\sigma_h,\tau_h) + c(u_h,\tau_h) = 0, \\ \qquad c(v_h,\sigma_h) + a(u_h,v_h) + b(v_h,p_h) = f(v_h), \\ \qquad b(u_h,q_h) = 0, \end{cases} \tag{5}$$

for which error estimates were derived in [7], [13]. We refer the reader to [3] for stabilized formulations of the mixed variational problem (4).

2.2. Block preconditioners

The linear system arising from (5) has the block form

$$Kx = \begin{pmatrix} D & C^{\mathrm{T}} & 0 \\ C & A & B^{\mathrm{T}} \\ 0 & B & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} D & \widetilde{C}^{\mathrm{T}} \\ \widetilde{C} & F \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \\ 0 \end{pmatrix} = g \tag{6}$$

and can be decoupled in various ways. A cheap alternative is offered by fixed-point solution methods such as Uzawa's method, see also [15] for another choice of fixed-point algorithm. Another possibility is pseudo-time-stepping or the Arrow-Hurwicz algorithm. Invariably, these methods trade convergence for iteration cost. For this reason we are interested in a Schur complement algorithm, which allows the decoupling of (6), but requires a good approximation of the Schur complement. Our aim is to derive an approximation for the Schur complement corresponding to the velocity and pressure variables. More precisely, we are interested in solving a preconditioned version of our problem

$$(KP^{-1})\hat{x} = g, \qquad \hat{x} = Px,$$

where

$$P = \begin{pmatrix} D & \widetilde{C}^{\mathrm{T}} \\ 0 & P_S \end{pmatrix}$$

with P_S an approximation to the Schur complement

$$S = F - \widetilde{C} D^{-1} \widetilde{C}^{\mathrm{T}}.$$

If $P_S = S$, the resulting preconditioned system KP^{-1} has only unit eigenvalues, so that convergence of an iterative solver in at most three steps [14]. In order to construct this approximation we make recourse to some of the existing theory regarding operator preconditioning for the case of elliptic systems of partial differential equations.

3. Schur complements for the EVSS problem

An incipient theory regarding preconditioning for elliptic systems is outlined in [4] as a generalization for the scalar case which was thoroughly analyzed in [5]. Further results regarding Schur complements were derived in [12]. We include below a brief outline which will be useful in describing the derivation of our preconditioner.

3.1. Preconditioners for elliptic systems of PDE

Consider the following system of PDE

$$\mathcal{L}(x, D)u(x) = f(x) \quad x \in \Omega, \tag{7a}$$

$$\mathcal{B}(x, D)u(x) = g(x) \quad x \in \Gamma, \tag{7b}$$

where $\mathcal{L}, \mathcal{B}$ are matrix differential operators of sizes $N \times N$, $m \times N$ respectively, with smooth coefficients. Definition of ellipticity is given below in the sense of Douglis and Nirenberg [1].

Definition 1. DN ellipticity. *A matrix operator $\mathcal{L}(x, D)$ with real coefficients is said to be elliptic if the following conditions hold:*
(i) *there exist integers $s_i \leq 0$ with $\max_i s_i = 0$ and $t_j > 0$ such that*

$$\deg(\mathcal{L}_{ij}(x,\xi)) \leq s_i + t_j, \quad i, j = 1, \ldots, N;$$

(ii) *the principal part of $\mathcal{L}$, denoted by $\mathcal{L}^0$ and defined as the sum of terms of $\mathcal{L}(x,\xi)$ homogeneous in ξ of degree $s_i + t_j$, satisfies*

$$\chi(x,\xi) = \det(\mathcal{L}^0(x,\xi)) \neq 0 \quad \forall\, x \in \Omega, \xi \in \mathbb{R}^n \setminus \{\mathbf{0}\};$$

If $n = 2$ we also require
(iii) *the polynomial $\chi(x, \xi + z\xi')$ in $z \in \mathbb{C}$ has exactly m roots with positive imaginary part for all linearly independent $\xi, \xi' \in \mathbb{R}^2$.*
We say the operator $\mathcal{L}$ has order

$$m = \frac{1}{2}\deg(\chi(x,\xi)) = \frac{1}{2}\sum_i s_i + t_i > 0$$

and DN numbers $(\mathbf{s}, \mathbf{t})$, where $\mathbf{s} = (s_1, ..., s_N), \mathbf{t} = (t_1, ..., t_N)$.

A regular elliptic system $(\mathcal{L}, \mathcal{B})$ requires $\mathcal{L}$ to be uniformly elliptic and $\mathcal{B}$ to satisfy a *complementing boundary condition* which guarantees the well-posedness of (7). This condition is defined below.

Definition 2. Complementing boundary condition. *Let $\mathcal{L}$ be an elliptic operator with DN numbers $(\mathbf{s}, \mathbf{t})$. A matrix differential operator $\mathcal{B}$ is said to fulfil the complementing boundary condition for $\mathcal{L}$ if*
(i) *there exist integers r_k such that*

$$\deg(\mathcal{B}_{kj}(x,\xi)) \leq r_k + t_j, \quad k = 1, \ldots, m,$$

and such that $\mathcal{B}_{kj} = 0$ for $r_k + t_j < 0$.
(ii) *Let $x_0 \in \Gamma$ and let $(\xi_1, \ldots, \xi_{n-1}, t)$ denote a new coordinate system where t is the coordinate in the direction of the inward normal to Γ at x_0. Let $D_t = (\xi_1, \ldots, \xi_{n-1}, \mathrm{d}/\mathrm{d}t)$. The initial value problem with constant coefficients*

$$\mathcal{L}^0(x_0, D_t)w(t) \;=\; 0 \quad t > 0, \tag{8a}$$

$$\mathcal{B}^0(x_0, D_t)w(t) \;=\; 0 \quad t = 0, \tag{8b}$$

has a unique solution satisfying $\lim_{t\to\infty} w(t) = 0$, which is the trivial solution. We say $\mathcal{B}$ has DN numbers $(\mathbf{t}, \mathbf{r})$ where $\mathbf{t}$ is as above and $\mathbf{r} = (r_1, \ldots, r_m)$.

Definition 3. *The system $(\mathcal{L}, \mathcal{B})$ is a regular elliptic system of order m with DN numbers $(\mathbf{s}, \mathbf{t}, \mathbf{r})$ if*
(i) *$\mathcal{L}$ is a uniformly elliptic operator of order m with DN numbers $(\mathbf{s}, \mathbf{t})$.*
(ii) *$\mathcal{B}$ satisfies the complementing boundary condition for $\mathcal{L}$ and has DN numbers $(\mathbf{t}, \mathbf{r})$.*

It is known (see [9], [17]) that a regular elliptic operator $\mathcal{K} = (\mathcal{L}, \mathcal{B}) : V \to W$ is a Fredholm operator for which there holds
(i) $\dim \ker(\mathcal{K}) < \infty$;
(ii) $\dim \operatorname{coker}(\mathcal{K}) = \dim(W/\operatorname{im}(\mathcal{K})) < \infty$,
where $\ker(\mathcal{K}), \operatorname{coker}(\mathcal{K})$ and $\operatorname{im}(\mathcal{K})$ are the kernel, cokernel and the image of $\mathcal{K}$, respectively. The index of a Fredholm operator is

$$\operatorname{index}(\mathcal{K}) = \dim \ker(\mathcal{K}) - \dim \operatorname{coker}(\mathcal{K}).$$

The following general result regarding preconditioning for elliptic systems can be found in [4].

Theorem 1. *Let $\mathcal{K} : V/\ker(\mathcal{K}) \to \operatorname{im}(\mathcal{K})$ be a regular elliptic operator with DN numbers $(\mathbf{s}, \mathbf{t}, \mathbf{r})$. If the regular elliptic operator $\mathcal{P}$ is invertible as a map from $V/\ker(\mathcal{K})$ to $\operatorname{im}(\mathcal{K})$ and has the same DN numbers and index as $\mathcal{K}$ then for some $\gamma < \infty$*

$$\|\mathcal{P}^{-1}\mathcal{K}\|_{V/\ker(\mathcal{K})} < \gamma, \quad \|\mathcal{K}^{-1}\mathcal{P}\|_{V/\ker(\mathcal{K})} < \gamma,$$
$$\|\mathcal{K}\mathcal{P}^{-1}\|_W < \gamma, \quad \|\mathcal{P}\mathcal{K}^{-1}\|_W < \gamma.$$

Remark 1. *Uniformly convergent discretizations of operators $\mathcal{K}, \mathcal{P}$ preserve the above equivalence. Thus, the above result is useful for designing preconditioners which are optimal with respect to the meshsize (performance of iterative solver is independent of the size of the problem).*

The following result found in [12] confirms in the sense of the above theorem that the Schur complement can be employed for preconditioning purposes.

Theorem 2. *Let $\mathcal{L}(D)$ be a $N \times N$ elliptic matrix differential operator with DN numbers $(\mathbf{s}, \mathbf{t})$ and assume that*

$$\mathcal{L}(D) = \begin{pmatrix} A(D) & B_1^{\mathrm{T}}(D) \\ B_2(D) & C(D) \end{pmatrix}$$

where $A(D)$ be an elliptic operator such that

$$\deg \det[A(\xi)]_{kl} = \deg \det[A^0(\xi)]_{kl}, \qquad \forall k, l, \tag{9}$$

where $[A(\xi)]_{kl}$ denotes the (kl)-cofactor of $A(\xi)$. Let $S(D)$ be a pseudo-differential operator defined by

$$S(D)\phi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left(C(\xi) - B_2(\xi)A(\xi)^{-1}B_1^t(\xi)\right) \hat{\phi}(\xi) \mathrm{e}^{-\mathrm{i}\xi\cdot x} \mathrm{d}\xi. \tag{10}$$

Then

$$\mathcal{P} = \begin{pmatrix} A(D) & B_1^{\mathrm{T}}(D) \\ 0 & S(D) \end{pmatrix}$$

is an elliptic operator with DN numbers $(\mathbf{s}, \mathbf{t})$.

The above result allows us to construct an approximation of $S(D)$ using (10) and therefore a useful preconditioner in the sense of Thm 1, provided we can find suitable boundary conditions. We remark here that one of the limitations of this approach is the requirement that our matrix differential operators have constant coefficients. We will see in the next section that numerically this is not an issue.

3.2. Preconditioners for the EVSS system

We now turn our attention to our application for the case when $n = 2$. To illustrate our approach, we first consider the simpler case of the three-field Stokes problem which corresponds to $\lambda = 0$ in (3). The symbols are

$$\mathcal{L}(\xi) = \begin{pmatrix} D(\xi) & C(\xi)^{\mathrm{T}} & 0 \\ C(\xi) & A(\xi) & B^{\mathrm{T}}(\xi) \\ 0 & B(\xi) & 0 \end{pmatrix}, \quad \mathcal{B}(\xi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

where

$$D(\xi) = \frac{1}{2\eta_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = -\begin{pmatrix} \xi_1 & \xi_2 & 0 \\ 0 & \xi_1 & \xi_2 \end{pmatrix}$$

$$A_0(\xi, \eta_2) = \begin{pmatrix} -\eta_2(2\xi_1^2 + \xi_2^2) & -\eta_2\xi_1\xi_2 \\ -\eta_2\xi_1\xi_2 & -\eta_2(\xi_1^2 + 2\xi_2^2) \end{pmatrix}, \quad B(\xi) = \begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix}.$$

The kernel of $(\mathcal{L}, \mathcal{B})$ has dimension one (as the pressure is defined up to a constant), but the codimension of the range is also equal to one (since the solution exists by imposing one constraint), so that $(\mathcal{L}, \mathcal{B}) = 0$. The corresponding DN numbers are found to be

$$\begin{aligned} \mathbf{s} &= (-1, -1, -1, 0, 0, -1), \\ \mathbf{t} &= (1, 1, 1, 2, 2, 1), \\ \mathbf{r} &= \begin{cases} (-2, -2) & \text{on } \Gamma \setminus \Gamma_{in} \\ (-1, -1) & \text{on } \Gamma_{in}. \end{cases} \end{aligned}$$

Thus, the system $(\mathcal{L}, \mathcal{B})$ represents an elliptic system of order 2. Consider now the symbol $F_0(\xi, \eta)$ of the Stokes operator with viscosity parameter η

$$F_0(\xi, \eta) = \begin{pmatrix} A_0(\xi, \eta) & B(\xi)^{\mathrm{T}} \\ B(\xi) & 0 \end{pmatrix}.$$

The symbol $\mathcal{L}(\xi)$ can thus be written as

$$\mathcal{L}(\xi) = \begin{pmatrix} D(\xi) & \widetilde{C}(\xi)^{\mathrm{T}} \\ \widetilde{C}(\xi) & F_0(\xi, \eta_2) \end{pmatrix}.$$

With the above notation, the symbol of the Schur complement corresponding to the velocity and pressure is found to be

$$S(\xi) = F_0(\xi, \eta_2) - \widetilde{C}(\xi) D(\xi)^{-1} \widetilde{C}(\xi)^{\mathrm{T}} = F_0(\xi, \eta_1 + \eta_2).$$

Consider therefore a preconditioning operator with symbol

$$P(\xi) = \begin{pmatrix} D(\xi) & \widetilde{C}(\xi)^{\mathrm{T}} \\ 0 & F_0(\xi, \eta_1 + \eta_2) \end{pmatrix}$$

Then $(\mathcal{P}, \mathcal{B})$ is an elliptic system with the same DN numbers as $(\mathcal{L}, \mathcal{B})$ and can immediately be seen to have index zero, so that Thm 1 applies. Note that the implementation of the preconditioner is done in two steps: we obtain the velocity and pressure by solving a Stokes system with viscosity parameter $\eta_1 + \eta_2$ and update the values of σ (since $D(\xi) = D$ is not a differential operator).

Consider now the linearized problem (3). Assuming constant coefficients, the resulting symbol has a similar form as in the Stokes case, except for the matrices D, A which now have the form

$$D_\lambda(\xi) = \frac{1 + \lambda b \cdot \xi}{\eta_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_b(\xi, \eta_2) = \begin{pmatrix} -\eta_2(2\xi_1^2 + \xi_2^2) + b \cdot \xi & -\eta_2 \xi_1 \xi_2 \\ -\eta_2 \xi_1 \xi_2 & -\eta_2(\xi_1^2 + 2\xi_2^2) + b \cdot \xi \end{pmatrix}$$

Assuming an explicit treatment of $B(b, \sigma)$ in (3), the resulting symbol is

$$\mathcal{L}(\xi) = \begin{pmatrix} D_0(\xi) & C(\xi)^{\mathrm{T}} & 0 \\ C(\xi) & A_b(\xi) & B^{\mathrm{T}}(\xi) \\ 0 & B(\xi) & 0 \end{pmatrix} = \begin{pmatrix} D_0(\xi) & \widetilde{C}(\xi)^{\mathrm{T}} \\ \widetilde{C}(\xi) & F_b(\xi, \eta_2) \end{pmatrix},$$

where $F_b(\xi, \eta_2)$ is the symbol of an Oseen operator

$$F_b(\xi, \eta_2) = \begin{pmatrix} A_b(\xi, \eta_2) & B(\xi)^{\mathrm{T}} \\ B(\xi) & 0 \end{pmatrix}.$$

As in the three-field Stokes case, the Schur complement for the velocity and pressure variables is found this time to be the symbol of an Oseen operator

$$S(\xi) = F_b(\xi, \eta_1 + \eta_2).$$

As before, the preconditioner

$$P(\xi) = \begin{pmatrix} D_0(\xi) & \widetilde{C}(\xi)^{\mathrm{T}} \\ 0 & F_b(\xi, \eta_1 + \eta_2) \end{pmatrix}$$

together with $\mathcal{B}$ is seen to form an elliptic system with the same DN numbers as $(\mathcal{L}, \mathcal{B})$.

4. Implementation

Any preconditioned iterative routine requires the action of the inverse of the preconditioner at every step. In our case, this is expressed by the following factorization of the block upper triangular candidate we propose

$$P^{-1} = \begin{pmatrix} D_0^{-1} & \\ & I \end{pmatrix} \begin{pmatrix} I & \widetilde{C}^{\mathrm{T}} \\ & I \end{pmatrix} \begin{pmatrix} I & \\ & F_b^{-1} \end{pmatrix}.$$

Thus, we need to invert an Oseen operator at every step, together with the constant operator D_0. The former operation can be achieved in several ways. Indeed, there is a certain amount of literature dedicated to this task. We highlight here a Schur complement approach that falls in the framework described in this work (see also [12]).

The operator F_b together with Dirichlet boundary conditions forms an elliptic system. The symbols are

$$F_b(\xi, \eta) = \begin{pmatrix} A_b(\xi, \eta) & B(\xi)^{\mathrm{T}} \\ B(\xi) & 0 \end{pmatrix}, \quad \mathcal{B}_F = (I_2 \ 0)$$

with DN numbers

$$\mathbf{s} = (0, 0, -1), \quad \mathbf{t} = (2, 2, 1), \quad \mathbf{r} = (-2, -2).$$

Following the examples above, one may consider the Schur complement corresponding to the pressure variables. The general form of the preconditioner is as before

$$P_F = \begin{pmatrix} F_b & B^{\mathrm{T}} \\ 0 & P_S \end{pmatrix}.$$

For the Oseen operator in the previous section the pressure Schur complement is found to have symbol

$$S(\xi) = \frac{-(\eta_1 + \eta_2)\xi^2 + b \cdot \xi}{-\xi^2}.$$

This is a pseudo-differential operator of order zero, so that we expect not to have to implement any boundary conditions. Indeed, the requirement

$$\deg \mathcal{B}_P(\xi) \leq \mathbf{r}_k + \mathbf{t}_j = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

indicates that the boundary operator corresponding to the pressure variable is zero (by definition of the complementing boundary condition). It follows that the choice $\mathcal{B}_P = \mathcal{B}_F$ yields an elliptic system $(P_F, \mathcal{B}_F)$ with the same DN numbers as $(F_b, \mathcal{B}_F)$. The implementation of P_F requires the inversion of F_b, which is an n-dimensional convection-diffusion operator, together with a discrete approximation of $S(\xi)$. A useful choice suggested in the literature is given by

$$P_S^{-1} = M_p^{-1} F_p A_p^{-1}$$

where M_p, F_p, A_p are mass, convection-diffusion and laplacian operators acting on the pressure space. Finite element assembly of these operators cannot implement the no-boundary-condition requirement; instead, Neumann boundary conditions were found to be sufficiently performant in the sense that the number of iterations is independent of the mesh-size.

5. Summary

It is evident that the approach presented here can be generalized to other systems of PDE. In the case of EVSS discretizations of viscoelastic flow models, our Schur complement technique allows essentially the decoupling of the problem into sub-problems corresponding to elastic stress variables and velocity-pressure variables. The latter is in general a challenging task – we discussed briefly a method that can be employed and is known to be optimal with respect to the mesh-size. Numerical experiments which are presented elsewhere [11] confirm that the resulting EVSS preconditioners exhibit the expected mesh independence predicted by the theory.

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