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Matlab Evaluation of the $\Omega_{j,k}^{m,n}(x)$ Large Coefficients for PDE Solving by Wavelet -Galerkin Approximation

by
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Abstract

This paper is one of a set of articles dealing with solutions to PDEs or ODEs using the wavelet - Galerkin method. In order to approximate the solution, a couple of families of coefficients are need; they occur in wavelet series and the are involved in discretizing differential equations that represent mathematical-mechanical models. Following some earlier ideas (see Reference list), we have achieved several algorithms and MATLAB - based programs allowing to obtain high precision results for the necessary functionals. Here it is described the MATLAB evaluation of the integral

$$\Omega_{j,k}^{m,n}(x) = \int_0^x \Phi(y)\,\Phi^{(m)}(y-j)\,\Phi^{(n)}(y-k)\,dy.$$

1 Introduction

Using some results due to Prof. Ingrid Daubechies (Princeton University, USA) regarding the determination of an orthonormal basis of functions with compact support on $L^2(\mathbb{R})$ [1], the team led by Prof. Chen (National Cheng Kung University of Taiwan) has proposed in [2] some algorithms for calculating seven functionals that occur in wavelet - Galerkin discretization of differential equations. In our paper we present the algorithms and programs needed for the calculation of one of these functionals, namely

$$\Omega_{j,k}^{m,n}(x) = \int_0^x \Phi(y)\,\Phi^{(m)}(y-j)\,\Phi^{(n)}(y-k)\,dy. \tag{1}$$

using the programming environment MATLAB. In expression (1), $j,k \in \mathbb{Z}$, $m,n \in \mathbb{N}^*$ and $\Phi^{(n)}(u)$ denotes the n-order derivative of function Φ. We will calculate the coefficients $\Omega_{j,k}^{m,n}$ (5).

2 Calculation of coefficients $\Omega_{j,k}^{m,n}$

Each member of the family of wavelets built by Daubechies is governed by a set of L (an integer number) coefficients $\{p_k : k = 0,1,\ldots,L-1\}$ and two functions $\Phi(x)$ and $\Psi(x)$.

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The function $\Phi(x)$, called the scalar or waveleted function is defined on $[0, L-1]$ and it has the expression

$$\Phi(x) = \sum_{j=0}^{L-1} p_j \Phi(2x - j).$$

The function $\Psi(x)$, called wavelet-mother, is defined on $[1 - L/2, L/2]$ and its expression is

$$\Psi(x) = \sum_{j=2-L}^{1} (-1)^j p_{1-j} \Phi(2x - j).$$

The Daubechies filtration coefficients p_k, $k = \overline{0, L-1}$ for $L = 6$ are the following:

$$p_0 = \frac{1 + \sqrt{10} + \sqrt{5 + 2\sqrt{10}}}{16}, \qquad p_1 = \frac{1 + \sqrt{10} + 3\sqrt{5 - 2\sqrt{10}}}{16},$$
$$p_2 = \frac{10 - 2\sqrt{10} + 2\sqrt{5 + 2\sqrt{10}}}{16}, \qquad p_3 = \frac{10 - 2\sqrt{10} - 2\sqrt{5 + 2\sqrt{10}}}{16},$$
$$p_4 = \frac{5 + \sqrt{10} - 3\sqrt{5 + 2\sqrt{10}}}{16}, \qquad p_5 = \frac{1 + \sqrt{10} - \sqrt{5 + 2\sqrt{10}}}{16}.$$

It can be seen that the equation $\sum_{k=0}^{5} p_k = 2$ is satisfied. We are going to calculated $\Omega_{j,k}^{m,n}(x)$ for $x = 5$, $L = 6$, $m = 0$, $n = 1$ and $-4 \le j, k \le 4$. $\Omega_{j,k}^{m,n}(x)$ plays an important role in the numerical solution of nonlinear differential equations by the wavelet - Galerkin method, according to the assertion and the example given by A. Latto and E. Tenenbaum, "Les ondelettes a support compact et la solution numerique de l'equation de Burgers", C. R. Acad. Sci. Paris, 311, 903-909 (1990). In the paper "The evaluation of connection coefficients of compactly supported wavelets" authored by A. Latto, H. L. Resnikoff and E. Tenenbaum and published in Proc. French - USA Workshop on wavelets and Turbulence, Y. Maday (ed.), the coefficient $\Omega_{j,k}^{m,n}(x)$ is called the third coefficient of wavelets connection.

The coefficients $\Omega_{j,k}^{m,n}(x)$ have the following properties:

$$\Omega_{j,k}^{m,n}(x) = 0 \quad \text{for} \quad |j|,\ |k|,\ \text{or}\ |j-k| \ge L-1, \tag{2}$$
$$\Omega_{j,k}^{m,n}(x) = 0 \quad \text{for} \quad x-j,\ x-k,\ \text{or}\ x \le 0, \tag{3}$$
$$\Omega_{j,k}^{m,n}(x) = \Omega_{j,k}^{m,n}(L-1) \quad \text{for} \quad x-j,\ x-k,\ \text{or}\ x \ge L-1. \tag{4}$$

In equation (113) of [2] we will take $-4 \le j, k \le 4$ and, taking into account formulas $(2)-(4)$ we will obtain a homogeneous system in the unknowns $\Omega_{j,k}^{m,n}(x)$ with $m = 0$ and $n = 1$. Equation (113) of [2] has the forms

$$\Omega_{j,k}^{m,n}(x) = 2^{m+n-1} \sum_{i_a=0}^{L-1} \sum_{i_b=0}^{L-1} \sum_{i_c=0}^{L-1} p_{i_a} p_{i_b} p_{i_c} \Omega_{2j+i_b-i_a,2k+i_c-i_a}^{m,n}(2x - i_a). \tag{5}$$

We have a system with $3L^2-9L+7$ unknowns $\Omega^{m,n}_{j,k}(L-1)$. We obtain, from equation (5) a homogeneous system of the form

$$v = 2^{1-m-n} S v. \tag{6}$$

where

$$v = [v_{2-L},\ v_{3-L},\ \ldots,\ v_{L-2}]^T, \tag{7}$$

$$v_j = \left[\Omega^{m,n}_{j,\alpha}(L-1),\ \Omega^{m,n}_{j,\alpha+1}(L-1),\ \ldots,\ \Omega^{m,n}_{j,\beta}(L-1)\right], \tag{8}$$

$\alpha = \max(j+2-L, 2-L)$, $\beta = \min(j+L-2, L-2)$, and the entries of matrix S are sums of products of the form $p_{i_a}p_{i_b}p_{i_c}$. Since $x=5$, $m=0$ and $n=1$, we denote the unknown by $\Omega_{j,k}$.
The unknowns of system are: $\Omega_{-4,-4}$; $\Omega_{-4,-3}$; $\Omega_{-4,-2}$; $\Omega_{-4,-1}$; $\Omega_{-4,0}$; $\Omega_{-3,-4}$; $\Omega_{-3,-3}$; $\Omega_{-3,-2}$; $\Omega_{-3,-1}$; $\Omega_{-3,0}$; $\Omega_{-3,1}$; $\Omega_{-2,-4}$; $\Omega_{-2,-3}$; $\Omega_{-2,-2}$; $\Omega_{-2,-1}$; $\Omega_{-2,0}$; $\Omega_{-2,1}$; $\Omega_{-2,2}$; $\Omega_{-1,-4}$; $\Omega_{-1,-3}$; $\Omega_{-1,-2}$; $\Omega_{-1,-1}$; $\Omega_{-1,0}$; $\Omega_{-1,1}$; $\Omega_{-1,2}$; $\Omega_{-1,3}$; $\Omega_{0,-4}$; $\Omega_{0,-3}$; $\Omega_{0,-2}$; $\Omega_{0,-1}$; $\Omega_{0,0}$; $\Omega_{0,1}$; $\Omega_{0,2}$; $\Omega_{0,3}$; $\Omega_{0,4}$; $\Omega_{1,-3}$; $\Omega_{1,-2}$; $\Omega_{1,-1}$; $\Omega_{1,0}$; $\Omega_{1,1}$; $\Omega_{1,2}$; $\Omega_{1,3}$; $\Omega_{1,4}$; $\Omega_{2,-2}$; $\Omega_{2,-1}$; $\Omega_{2,-0}$; $\Omega_{2,1}$; $\Omega_{2,2}$; $\Omega_{2,3}$; $\Omega_{2,4}$; $\Omega_{3,-1}$; $\Omega_{3,0}$; $\Omega_{3,1}$; $\Omega_{3,2}$; $\Omega_{3,3}$; $\Omega_{3,4}$; $\Omega_{4,0}$; $\Omega_{4,1}$; $\Omega_{4,2}$; $\Omega_{4,3}$; $\Omega_{4,4}$.
Taking $j=-4$ and $k=-4$ in equation (5) and taking into account (2) – (4) we obtain

$$\Omega_{-4,-4} = (p_0p_4p_4 + p_1p_5p_5)\,\Omega_{-4,-4} + p_0p_4p_5\Omega_{-4,-3} + p_0p_5p_4\Omega_{-3,-4} + p_0p_5p_5\Omega_{-3,-3}.$$

It follows that

$$s_{11} = p_0p_4p_4 + p_1p_5p_5,\quad s_{12} = p_0p_4p_5,\quad s_{13} = p_0p_5p_4,\quad s_{14} = p_0p_5p_5;$$

the remaining entries on the first row being equal to zero.
Similarly, if we consider $j=-4$ and $k=-3$ in formula (5) we have

$$\begin{aligned}\Omega_{-4,-3} = {} & (p_0p_4p_2 + p_1p_5p_3)\,\Omega_{-4,-4} + (p_1p_5p_4 + p_0p_4p_3)\,\Omega_{-4,-3} + \\ & + (p_0p_4p_4 + p_1p_5p_5)\,\Omega_{-4,-2} + p_0p_4p_5\Omega_{-4,-1} + p_0p_5p_2\Omega_{-3,-4} + \\ & + p_0p_5p_3\Omega_{-3,-3} + p_0p_5p_4\Omega_{-3,-2} + p_0p_5p_5\Omega_{-3,-1}.\end{aligned}$$

It follows that

$$\begin{gathered} s_{21} = p_0p_4p_2 + p_1p_5p_3,\quad s_{22} = p_1p_5p_4 + p_0p_4p_3,\\ s_{23} = p_0p_4p_4 + p_1p_5p_5,\quad s_{24} = p_0p_4p_5,\quad s_{25} = 0,\\ s_{26} = p_0p_5p_2,\quad s_{27} = p_0p_5p_3,\quad s_{28} = p_0p_5p_4,\quad s_{29} = p_0p_5p_5; \end{gathered}$$

the remaining entries in the second row are $=0$. Following this procedure, the matrix S is generated:

```
% The generate matrix omega
clc
p1=0.47046720778416;
p2=1.14111691583144;
p3=0.65036500052623;
```

```
p4=-0.19093441556833;
p5=-0.12083220831040;
p6=-0.04981749973688;
a=[-4 -4 -4 -4 -4 -3 -3 -3 -3 -3 -3 -2 -2 -2 -2 -2 -2 -2 -1 ...
-1 -1 -1 -1 -1 -1 -1 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 2 2 2 ...
2 2 2 2 3 3 3 3 3 3 4 4 4 4 4];
b=[-4 -3 -2 -1 0 -4 -3 -2 -1 0 1 -4 -3 -2 -1 0 1 2 -4 -3 -2 ...
-1 0 1 2 3 -4 -3 -2 -1 0 1 2 3 4 -3 -2 -1 0 1 2 3 4 -2 -1 0 ...
1 2 3 4 -1 0 1 2 3 4 0 1 2 3 4];
L=6;
s=zeros(61);
for r=1:61
    j=a(r);
    k=b(r);
    for ia=1:L
        for ib=1:L
            for ic=1:L
                jj=2*j+ib-ia;
                kk=2*k+ic-ia;
                for t=1:61
                    if ((jj==a(t))&(kk==b(t))
                        q=t;
                        s(r,q)=s(r,q)+p(ia)*p(ib)*p(ic);
                        break
                    end
                end
            end
        end
    end
end
[vp,dp]=eig(s)
```

The matrix S has the eigenvalues 2^{1-k}, $k = 0, 1, \ldots, L-2$ with the multiplicity order $k+1$.
It can be seen from (6) that v is an eigenvector corresponding to the eigenvalue 2^{m+n-1}. In our case v is the solution corresponding to the eigenvalue 1 with the multiplicity order two. It follows that we cannot determine a unique solution from (6).
In order to determine a solution to system (6) we attach the equations resulting from the equation of

moments (formulas (117) and (118) of [2]), namely

$$
\begin{aligned}
&-4\Omega_{-4,-4} - 3\Omega_{-4,-3} - 2\Omega_{-4,-2} - 1\cdot\Omega_{-4,-1} + 0\cdot\Omega_{-4,0} && = \Gamma^0_{-4}\\
&-4\Omega_{-3,-4} - 3\Omega_{-3,-3} - 2\Omega_{-3,-2} - 1\cdot\Omega_{-3,-1} + 0\cdot\Omega_{-3,0} + 1\cdot\Omega_{-3,1} && = \Gamma^0_{-3}\\
&-4\Omega_{-2,-4} - 3\Omega_{-2,-3} - 2\Omega_{-2,-2} - 1\cdot\Omega_{-2,-1} + 0\cdot\Omega_{-2,0} + 1\cdot\Omega_{-2,1} +\\
&+2\Omega_{-2,2} && = \Gamma^0_{-2}\\
&-4\Omega_{-1,-4} - 3\Omega_{-1,-3} - 2\Omega_{-1,-2} - 1\cdot\Omega_{-1,-1} + 0\cdot\Omega_{-1,0} + 1\cdot\Omega_{-1,1} +\\
&+2\Omega_{-1,2} + 3\Omega_{-1,3} && = \Gamma^0_{-1}\\
&-4\Omega_{0,-4} - 3\Omega_{0,-3} - 2\Omega_{0,-2} - 1\cdot\Omega_{0,-1} + 0\cdot\Omega_{0,0} + 1\cdot\Omega_{0,1} + 2\Omega_{0,2} +\\
&+3\Omega_{0,3} + 4\Omega_{0,4} && = \Gamma^0_{0}\\
&-3\Omega_{1,-3} - 2\Omega_{1,-2} - 1\cdot\Omega_{1,-1} + 0\cdot\Omega_{1,0} + 1\cdot\Omega_{1,1} + 2\Omega_{1,2} + 3\Omega_{1,3} +\\
&+4\Omega_{1,4} && = \Gamma^0_{1}\\
&-2\Omega_{2,-2} - 1\cdot\Omega_{2,-1} + 0\cdot\Omega_{2,0} + 1\cdot\Omega_{2,1} + 2\Omega_{2,2} + 3\Omega_{2,3} + 4\Omega_{2,4} && = \Gamma^0_{2}\\
&-1\cdot\Omega_{3,-1} + 0\cdot\Omega_{3,0} + 1\cdot\Omega_{3,1} + 2\Omega_{3,2} + 3\Omega_{3,3} + 4\Omega_{3,4} && = \Gamma^0_{3}\\
&0\cdot\Omega_{4,0} + 1\cdot\Omega_{4,1} + 2\Omega_{4,2} + 3\Omega_{4,3} + 4\Omega_{4,4} && = \Gamma^0_{4}\\
&\Omega_{-4,-4} + \Omega_{-3,-4} + \Omega_{-2,-4} + \Omega_{-1,-4} + \Omega_{0,-4} && = \Gamma^1_{-4}\\
&\Omega_{-4,-3} + \Omega_{-3,-3} + \Omega_{-2,-3} + \Omega_{-1,-3} + \Omega_{0,-3} + \Omega_{1,-3} && = \Gamma^1_{-3}\\
&\Omega_{-4,-2} + \Omega_{-3,-2} + \Omega_{-2,-2} + \Omega_{-1,-2} + \Omega_{0,-2} + \Omega_{1,-2} + \Omega_{2,-2} && = \Gamma^1_{-2}\\
&\Omega_{-4,-1} + \Omega_{-3,-1} + \Omega_{-2,-1} + \Omega_{-1,-1} + \Omega_{0,-1} + \Omega_{1,-1} + \Omega_{2,-1} + \Omega_{3,-1} && = \Gamma^1_{-1}\\
&\Omega_{-4,0} + \Omega_{-3,0} + \Omega_{-2,0} + \Omega_{-1,0} + \Omega_{0,0} + \Omega_{1,0} + \Omega_{2,0} + \Omega_{3,0} + \Omega_{4,0} && = \Gamma^1_{0}
\end{aligned}
$$

$$
\begin{aligned}
&\Omega_{-3,1} + \Omega_{-2,1} + \Omega_{-1,1} + \Omega_{0,1} + \Omega_{1,1} + \Omega_{2,1} + \Omega_{3,1} + \Omega_{4,1} && = \Gamma^1_{1}\\
&\Omega_{-2,2} + \Omega_{-1,2} + \Omega_{0,2} + \Omega_{1,2} + \Omega_{2,2} + \Omega_{3,2} + \Omega_{4,2} && = \Gamma^1_{2}\\
&\Omega_{-1,3} + \Omega_{0,3} + \Omega_{1,3} + \Omega_{2,3} + \Omega_{3,3} + \Omega_{4,3} && = \Gamma^1_{3}\\
&\Omega_{0,4} + \Omega_{1,4} + \Omega_{2,4} + \Omega_{3,4} + \Omega_{4,4} && = \Gamma^1_{4}
\end{aligned}
$$

The numbers $\Gamma^0_{-4}, \Gamma^0_{-3}, \Gamma^0_{-2}, \ldots, \Gamma^1_4$ are known.
The attachment of these equations to system (6) is accomplished after the elimination of the rows corresponding to the unknowns $\Omega_{-4,0}$, $\Omega_{-3,0}$ and $\Omega_{0,0}$. The replacement of the rather difficult; the obtained solution must satisfy the conditions (117) and (118) of [2].

```
% Program for determinate solution
for i=1:61
    for j=1:61
        if i==j
            s(i,j)=-1+s(i,j)
        end
    end
end
s(5,1:4)=[-4 -3 -2 -1]; s(5,5:61)=0;
```

```
s(10,1:5)=0; s(10,6:11)=[-4 -3 -2 -1 0 1]; s(10,12:61)=0;
% s(31,1:11)=0; s(31,12:18)=[-4 -3 -2 0 1 2]; s(31,20:61)=0;
rang=rank(s)
dets=det(s);
d=zeros(61,1);
d(5,1)=-0.34246575e-3;
d(10,1)=-0.14611872e-1;
% d(31,1)=0.14520548;
format long
sol=s\d
```

The solution thus obtained has a higher accuracy than the one of [2]

References

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