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Spline Approximation Techniques and Regularization Methods for First Kind Integral Equations

by

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Abstract

In this paper we realize a study on spline approximation method and different types of regularization techniques (like multilevel Landweber iteration, and a multilevel Tikhonov schemes with zero's order stabili- zers). All these methods are applied to several (linear) first kind integral Fredholm equations. Advantages of developed methods are proved by numerical experiments when compared to some standard techniques.

Keyword and phrases: first kind integral equation, linear least-squares problem, minimal norm solution, spline functions, regularization metods, projection, collocation.

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1 Introduction

Let $K : L^2([a, b]) \to L^2([a, b])$ be the (compact) integral operator

$$Kx(t) = \int_a^b k(t,s)x(s)ds, \tag{1}$$

and the equation

$$Kx(t) = y(t), \quad \forall\ t \in [0,1]. \tag{2}$$

with square-integrable kernel $k : [a, b] \times [a, b] \to I\!R$, and $y \in L^2([a, b])$. Our problem is to derive $x(s)$ when the data function $y(s)$ and the kernel are known exactly, or only approximately.

2 Spline Approximation Method

We shall start by briefly presenting the projection method used to solve the equation (2).
Let $n \geq 1$ be arbitrary fixed and $\{v_1, v_2, \dots, v_n\} \subseteq \overline{R(K)}$ a set of vectors with $\|v_i\| = 1,\ \forall\ i \in I\!N$. We consider the following discretization of the equation (2): find $x \in X_n$ such that

$$\langle Kx, v_i\rangle = \langle y, v_i\rangle,\ \forall\ i = 1, \dots n, \tag{3}$$

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where $X_n = span\{K^*v_1, \ldots, K^*v_n\}$. If, for any $n \geq 1$, the set $\{v_1, v_2, \ldots, v_n\} \subseteq \overline{R(K)}$ is linearly independent, then the discrete problem (3) has a unique solution $x_n \in X_n$ given by (see [6])

$$x_n = (K^*v_1, K^*v_2, \ldots, K^*v_n)Q_n^{-1}(\langle y, v_1\rangle\langle y, v_2\rangle, \ldots, \langle y, v_n\rangle)^t \tag{4}$$

or equivalent

$$x_n = \sum_{j=1}^{n} \alpha_j K^* v_j, \tag{5}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is the unique solution of the system

$$Q_n\alpha = b$$

whith $Q_n = (\langle K^*v_i, K^*v_j\rangle)_{i,j=\overline{1,n}}$, $b = (b_1, \ldots, b_n)^t \in I\!R^n$, $b_i = \langle y, v_i\rangle$. Since $K^+y = K^+P_{\overline{RK}}y$, the solution of (3) remains the same if y is replaced by $P_{\overline{RK}}y$. So, one can assume that $y \in R(K)$. The following result is proved in [6] (Theorem 6).

Theorem 1 *Under the above condition of linearly independency, and if $span\{v_1, \ldots, v_n, \ldots\}$ is dense in $\overline{R(K)}$, then $\lim\limits_{n\to\infty} x_n = x_{LS}$, where x_{LS} is the minimal norm solution of the least-square problem associated to (2).*

Remark 1 *In [7] it is proved that the previous theorem still holds even if we don't have the linear independent functions, but under a milder condition.*

The main idea presented in [2] is to use in the projection method using as v_i the spline functions. For this, let $a = x_1 < x_2 < \ldots < x_n = b$ a partition of $[a, b]$. That author requires that y is $b-a$- periodic function. This is not a constrained condition since we can define the other (eventually needed) values as $y(s_j) = y(s_{j+n}),\ j \leq 0$, and $y(s_j) = y(s_{j-n}),\ j > n$, and the knots x_j with $j < 0$ or $j > n$ are chosen on peridiocity. We shall denote by $s_{i,2m-1}(x)$ the local polynomial spline of $2m-1$ degree constructed on knots $x_i, \ldots, x_{i+2m},\quad i = -2m+1, \ldots, n-1$. The formulas for the local spline and the algorithms of their stable calculation is given in [1]. In our example we shall use the cubic spline polynomials (so, $m = 2$). Thus, the approximated minimal norm solution will be a linear combination of such splines.
In [2] it is proved that (Theorem 1.2.1.) any solution for the initial equation obtained by the collocation method is also solution obtained by the projection method. As in most cases, the first method is more tractable to deal with, we shall use this one in numerical experiments.

Remark 2 *Another way to approximate the minimal norm solution is using the trigonometric spline function.*

Remark 3 *As future work, we intend to use the spline approximation method in order to solve an integral equation for a single layer equation associated with Laplace Dirichlet problem.*

3 Regularization Methods

Even if we formulate (2) in the least-square sense, if K is of infinite rank, we have problems in solving it because the Moore-Penrose inverse $K^+ : D(K^+) = R(K) \oplus R(K)^{\perp} \to I\!R$ is unbounded, and as we have a noise in the data, namely

$$\parallel y - y_\delta \parallel \leq \delta, \tag{6}$$

one cannot expect the solution of the perturbed least-squares equation to be a good approximation to the exact least-squares $x_{LS} = K^+ y$. This is because by its very nature, the initial problem is ill-posed. In order to overcome this shortcoming, it is considered the regularized equation of the normal equation

$$K^* K x_\delta = K^* y_\delta, \tag{7}$$

where K^* is the adjoint of the operator K. Such (regularized) equations are computationally more tractable, but, in this case, another difficulty arises: to find a good regularization parameter. This task can be an expensive procedure. For example, for the standard Landweber iteration, for an n-point discretization of (2) it is required $2in^2$ operations, where i is the number of iterations, which can be quite large; also, for the Tikhonov method the cost is $\frac{n^3}{2} + \frac{jn^3}{6}$.
In what follows, we shall briefly present multilevel schemes which reduce the above mentioned computational cost (for details see [5]).

3.1 Auxiliary Results

For the compact operator K, let $\{u_n,\ v_n,\ \mu_n\}$ be the singular system given by the singular value decomposition theorem (for short, the SVD theorem): $\{v_n\}$ is the orthonormal eigenvector system for K^*K with the eigenvectors $\lambda_1^2 \geq \lambda_2^2 \geq \ldots$, $\mu_n = |\lambda_n|^{-1}$, and $u_n = \mu_n K v_n$. It is known that $\{v_n\}$, $\{u_n\}$ respectively form orthonormal bases in $\overline{R(K^*)}$, $\overline{R(K)}$ respectively. Also, the Picard Criteria for solvability and stability of (2) states the following (for more details and for proofs, see [3]).

Theorem 2 *Equation* (2) *has a solution if and only if*
(i) $y \in N(K^*)^{\perp}$, and
(ii) $\sum\limits_{n=1}^{\infty} \mu_n^2 \left|(y, u_n)\right|^2 < \infty$.
Under these assumptions, the solution is

$$x = \sum_{n=1}^{\infty} \mu_n (y, u_n) v_n. \tag{8}$$

Remark 4 *Problems appear when* y *is perturbed by* δy*, because, in this case, or for* $y + \delta y$ *the condition* (ii) *may not hold, or if it does, the series* $\sum\limits_{n=1}^{\infty} \mu_n (\delta y, u_n)$ *may be noticeable (as* $\mu_n \to \infty$*). This is because* $R(K)$ *is not closed (or* $\dim R(K) = \infty$*).*

Theorem 3 *If $y \in D(K^+)$, then the minimal norm solution (for exact data) is given*

$$x_{LS} = K^+ = \sum_{n=1}^{\infty} \mu_n (Py, u_n) v_n = \sum_{n=1}^{\infty} \mu_n (y, u_n) v_n, \tag{9}$$

where P is the orthogonal projector onto $\overline{R(K)}$.

Remark 5 *As in the previous remark, if $R(K)$ is not closed, the perturbed least-squares has the same instability problem.*

3.2 Landweber Iteration. Tikhonov Regularization

The aforementioned problems can be solved using the regularization algorithms (the main results can be found in [4]). The Landweber iteration and the Tikhonov regularization methods are defined as

$$x_{n+1}^\delta = x_n^\delta + \mu(K^* y_\delta - K^* K x_n^\delta),\ x_0^\delta = 0,\ 0 < \mu < \frac{2}{\| K^* K \|} = \frac{2}{\| K \|^2}, \tag{10}$$

and

$$x_{\alpha(\delta)}^\delta = [K^* K + \alpha(\delta)]^{-1} K^* y_\delta, \tag{11}$$

respectively, where $x_\alpha,\ x_\alpha^\delta$ are the solutions of the regularized equation with exact, and perturbed data respectively. The following estimations hold.

Theorem 4

$$\| K(x_\alpha - x_\alpha^\delta) \| \le \delta M, \quad \| x_\alpha - x_\alpha^\delta \| \le \delta \sqrt{M r(\alpha)}. \tag{12}$$

Remark 6 *For the Landweber-Fridman iteration, $M = 1,\ r(n) = \mu n$, and if $n(\delta)$ is chosen such that $\delta^2 \mu n(\delta) \to 0,\ \delta \to 0$, then $x_{n(\delta)}^\delta \to x_{LS}$; for the Tikhonov scheme, $M = 1,\ r(\alpha) = \frac{1}{\alpha}$, and if $\frac{\delta^2}{\alpha(\delta)} \to 0,\ \delta \to 0$, then $x_{\alpha(\delta)}^\delta \to x_{LS}$.*

The Morozov discrepancy principle chooses the unique $\alpha(\delta)$ with the property $\| K x_{\alpha(\delta)} - y_\delta = \delta \|$. For the first kind integral equation, the Landweber iteration is

$$x_n^\delta(s) = x_{n-1}^\delta(s) + \int_a^b k(v,s) \left[y_\delta(t) - \int_a^b k(v,t) x_{n-1}^\delta(t)\ \mathrm{dt} \right]\ \mathrm{dv},$$

equation which is solved after is discretized as

$$\tilde{x}_n^\delta = \tilde{x}_{n-1}^\delta + h K_{hh}^t \left[\tilde{y}_{\delta h} - h K_{hh} \tilde{x}_{n-1,h}^\delta \right],$$

where h is the step size of the discretization, and K_{hh} is the discretized kernel with stepsize h. The theory assures us (see [4], [5]) that both

$$\| x_n^\delta(s) - x_{LS} \| \to 0,\ \delta \to 0$$

and

$$\| \tilde{x}^{\delta}_{n,h} - x_{LS} \| \to 0, \ \delta \to 0,$$

and also, the quadrature error goes to 0. The idea of the multilevel schemes is to monotorize the residual; if the residual does not change much after a coarse-grid correction, then only additional Landweber iteration on the fine grid should be performed. In [5] is said that α should not to be too small to permit magnification of roundoff errors which can be obtained on a grid coarser than H. If this grid is $4h$, letting $H = 2h$, the number of operation is smaller than in the standard approach. The standard form of the Tikhonov scheme is

$$(K^*K + \alpha(\delta)I)x^{\delta}_{\alpha(\delta)} = K^* y_{\delta}. \tag{13}$$

The zero'th order stabilizer is $f(x) = \| x \|^2_{L^2}$ which applied to a first kind integral equation produces an integro-differential equation with boundary conditions as follows

$$\int_a^b \int_a^b k(v,s)k(v,t)x^{\delta}_{\alpha(\delta)}(t) \text{ dv dt} + \alpha(\delta) = \int_a^b k(v,s)y_{\delta}(v) \text{ dv},$$

with $x^{\delta}_{\alpha(\delta)}(a) = x_1, \ x^{\delta}_{\alpha(\delta)}(b) = x_2$. As a parameter choice is used quasi-optimal method i.e. $\alpha_k = \mu\alpha_{k-1}, \ 0 < \mu < 1$, and then chooses the parameter that minimizes $\| x^{\delta}_{\alpha_n(\delta)} - x^{\delta}_{\alpha_{n-1}(\delta)} \|$. The idea of the Tikhonov multilevel schemes consists in: using n levels, the coarsest level is solved using the discrepancy principle with Choleschy decomposition, and then the higher levels are solved using the discrepancy stopping criterion with an iterative system solver, and, thus, it is reducing the operations number visibly.

4 Numerical Experiments

Problem 1. Let the equation (derived from antenna design theory)

$$\int_{-\pi}^{\pi} \cos(st)x(t) \text{ dt} = 2\pi \left[S((1+s)\pi) + S((1-s)\pi) \right],$$

where $S(s) = \int_0^s \frac{\sin(u)}{u} \text{ du}$, and the solution $x(t) = 2\pi \frac{\sin(t)}{t}$. After we transformed this equations from $[-\pi, \pi]$ to $[0,1]$, discretize it, and using the values $h = \sqrt{\frac{12\delta}{\|K_{hh}\|}}$, α is chosen using the Morozov principle on the coarsest grid (stepsize $4h$), but $\alpha \geq 0.00005$ in order to prevent propagation of roundoff errors in the interpolation procedure, the noise is $tr(K^t_{hh} K_{hh})\delta$, the data are presented in the table 1 and 2.
Problem 2. Let the Phillip's equation

$$\int_{-3}^{3} k(t-s)x(t) \text{ dt} = y(s), \ s \in [-6, 6]$$

δ/h	iterations	$\| err \|_2$
0.0005/0.0078125	21	0.0324145
0.00025/0.0039063	95	0.0241030
0.0001/0.0019531	348	0.0103711

Table 1: Results obtained with a standard Landweber iteration

δ/h	α	$\| err \|_2$
0.0005/0.0078125	0.02293	0.0254669
0.00025/0.0039063	0.00717	0.0158831
0.0001/0.0019531	0.00250	0.00739896

Table 2: Results obtained with a multigrid Landweber iteration

where $k(u) = \begin{cases} 1+\cos(\pi u/3), & |u| \le 3 \\ 0, & |u| \ge 3, \end{cases}$ and

$y(s) = \begin{cases} (6-s)\left[1+\frac{1}{2}\cos(\frac{\pi s}{3})\right] + \frac{9}{2\pi}\sin(\frac{\pi s}{3}), & s \in [0,6] \\ (6+s)\left[1+\frac{1}{2}\cos(\frac{\pi s}{3})\right] - \frac{9}{2\pi}\sin(\frac{\pi s}{3}), & s \in [-6,0] \end{cases}$,

with the exact solution $x(t) = \begin{cases} 1+\cos(\pi t/3), & |t| \le 3 \\ 0, & |t| \ge 3. \end{cases}$

The initial value of α is 1, $\mu = 0.5$, a noise $y(s_j)\delta\theta_j$ where θ is a random number chosen from a uniform distribution on $[-1,1]$, and the generalized discrepancy principle. The results are those from Table 3 and 4.

δ/h	α	$\| err \|_2$
0.0002/0.015625	0.005722	0.0490729
0.0001/0.0078125	0.002576	0.0322052
0.00002/0.00390625	0.0009570	0.0233487

Table 3: Results obtained with a standard Tikhonov method with 0'th order stabilizer

δ/tol	α	$\| err \|_2$
$0.0002/10^{-6}$	0.003725	0.0391557
$0.0001/10^{-6}$	0.001572	0.0260117
$0.00002/10^{-8}$	0.0007053	0.0227495

Table 4: Results obtained with multigrid Tikhonov method

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