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A mathematical model for the strongly nonlinear saturated-unsaturated infiltration

by
GABRIELA MARINOSCHI[1,2]

Abstract

In this paper we set the mathematical model of a saturated-unsaturated water infiltration into a porous medium and we present an existence result for Richards' equation.

1 Physical context and mathematical hypotheses

From the hydraulic point of view, the problems we shall study are related to a Darcian flow of an incompressible fluid in an isotropic, homogeneous nondeformable porous medium with a constant porosity with no hysteresis development.

The general boundary value problem Assume that we have to study the water infiltration in a domain Ω, which is an open bounded subset of $\mathbf{R}^N$, $(N = 1, 2, 3)$, within the finite time interval $(0, T)$. The boundary of Ω is denoted by Γ and it is considered smooth enough, e.g., of class C^1.The vector of space variables is denoted by $x = (x_1, x_2, x_3)$ and the time by t.
We consider as basic Richards' equation, for an isotropic and homogeneous medium, with initial data and various boundary conditions,

$$\frac{\partial \theta}{\partial t} - \nabla \cdot (k(h)\nabla h) + \frac{\partial k(h)}{\partial x_3} = f \text{ in } Q = \Omega \times (0, T), \tag{1}$$

$$h(x, 0) = h_0(x) \text{ in } \Omega, \tag{2}$$

$$\textit{boundary conditions in } h \text{ on } \Sigma = \Gamma \times (0, T). \tag{3}$$

In particular we shall treat the case with flux-type boundary conditions

$$\mathbf{q} \cdot \nu = u(x, t) \text{ on } \Sigma_u = \Gamma_u \times (0, T), \tag{4}$$

$$\mathbf{q} \cdot \nu = \alpha(x)K^*(h) + f_0(x, t) \text{ on } \Sigma_\alpha = \Gamma_\alpha \times (0, T), \tag{5}$$

[1]Institute of Mathematical Statistics and Applied Mathematics, Calea 13 Septembrie no. 13, 050711 Bucharest, Romania
E-mail: gmarino@acad.ro
[2]This work was elaborate under the contract CEEX-05-D11-25/5.11.2005

where, by $\mathbf{q}$ we denoted the flux defined by

$$\mathbf{q}(x,t) = k(h)\mathbf{i}_3 - \nabla K^*(h), \tag{6}$$

the function $K^*(h)$ will be defined below, $\alpha : \Gamma_\alpha \to [\alpha_m, \alpha_M]$ is a continuous positive function, $\alpha_m > 0$, ν is the outer normal vector at the boundary and $\mathbf{i}_3$ is the unit vector of the Ox_3 axis, downwards directed.

Description of the hydraulic model We emphasize the fact that the behaviour of an unsaturated soil, i.e., partially filled with water, is completely known from the hydraulic point of view if two functions are given: one is the constitutive relationship

$$\theta := C^*(h), \tag{7}$$

linking the volumetric water content, or moisture of the soil θ, to the pressure head, h, and the other is the hydraulic conductivity

$$k := k(h), \tag{8}$$

both depending nonlinearly on h. For an isotropic soil the latter is a scalar function.
Since we study the nonhysteretic case, the constitutive law and the hydraulic conductivity are single-valued functions of pressure. We stress that in the unsaturated flow we denote by h the negative value of the capillary pressure.
Therefore, these functions are defined in the unsaturated flow for negative values of the unsaturated pressure between a minimum value, $h = h_r < 0$ and $h = 0$. The value h_r corresponds to the residual moisture θ_r specified as the quantity of water resident in soil and $h = 0$ is the pressure head value at which saturation is reached. Moreover, the value θ_r is related to the notion of field capacity which means, in other words, that infiltration may evolve from the field capacity up to the saturation value.
Correspondingly, the water capacity defined as the derivative of the moisture with respect the pressure

$$C(h) := \frac{d\theta}{dh}, \tag{9}$$

has a unique maximum at h_r.
For the saturated flow, when h becomes zero and then positive, the previously defined functions take constant values all over $[0, \infty)$. Now, h represents the saturated hydraulic pressure that increases as the water column increases.
We intend to show how that the particular character of the hydraulic models is determined by the behaviour of the functions C^* and k around 0.

Mathematical hypotheses For the unsaturated flow, where $h < 0$, we assume the following:
(m_1) $C^* : [h_r, 0) \to [\theta_r, \theta_s)$ is single-valued, positive, twice differentiable on $[h_r, 0)$, monotonically increasing and concave;
(m_2) $k : [h_r, 0) \to [K_r, K_s)$ is single-valued, positive, twice differentiable on $[h_r, 0)$, monotonically increasing and convex, satisfying the property $k'(h_r) = 0$;

(m_3) $C : [h_r, 0) \to (C_0, C_r]$ is single-valued, non-negative, differentiable on $[h_r, 0)$ monotonically decreasing and satisfies $C'(h_r) = 0$;
(m_4) there exist

$$\theta_s := \lim_{h \nearrow 0} (C^*)(h) > 0, \tag{10}$$

$$C_0 := \lim_{h \nearrow 0} (C^*)'(h) \geq 0, \tag{11}$$

$$K_s := \lim_{h \nearrow 0} k(h) > 0. \tag{12}$$

We denote

$$K_0' := \lim_{h \nearrow 0} k'(h), \ K_0' \in [0, \infty) \cup \{\infty\}. \tag{13}$$

In the saturated flow we have
(m_5) $C^*(h) = \theta_s$, $k(h) = K_s$ and $C(h) = 0$ for $h > 0$.
Therefore, we see that the unsaturated flow is characterized by $h < 0$ or by $\theta \in [\theta_r, \theta_s)$ while the saturated one by $h \geq 0$ or $\theta = \theta_s$.
The positive values θ_r, θ_s and their corresponding conductivities K_r, K_s are soil characteristics and are known. The properties $k'(h_r) = 0$ and $C'(h_r) = 0$ were put into evidence by experiments, (see [1]).
We notice that the functions C^* and k are continuous on $[h_r, \infty)$ and h_r is the maximum point for C and a saddle point for C^*. Also C is continuous on $[h_r, \infty)$, except possibly at the point 0.
We stress the fact that these properties are verified by the empiric hydraulic models set up in the last decades (see e.g., [1]).
In the following, we shall try to explain which analytical characteristics of these functions may lead to these possible values. We have noticed that the main role is played by the increase rate of the functions C^* and k around 0. The significant contribution is the behaviour of the constitutive law C^*, while the rate of k may determine a particular behaviour without equalizing however the main character imprint by C^*.

2 Strongly nonlinear saturated-unsaturated diffusive model

Let us assume $(m_1) - (m_5)$ and

$$C_0 = 0$$

which is the main characteristic of this case. It follows then that C is continuous on $[h_r, \infty)$ and so we can write $C^* : [h_r, \infty) \to [\theta_r, \theta_s]$, as

$$C^*(h) = \begin{cases} \theta_r + \int_{h_r}^{h} C(\zeta) d\zeta, \ h < 0, \\ \theta_s, \ h \geq 0. \end{cases} \tag{14}$$

Strongly nonlinear hydraulic conductivity This situation corresponds to $K_0' \in \mathbf{R}_+ = (0, \infty)$ or $K_0' = +\infty$.
We define an antiderivative of K by

$$K^*(h) := \begin{cases} K_r^* + \int_{h_r}^{h} k(\zeta)d\zeta, \ h < 0, \\ K_s^* + K_s h, \ h \geq 0, \end{cases} \tag{15}$$

where $K^* : [h_r, \infty) \to [K_r^*, K_s^*]$ and

$$K_s^* := K^*(0) > 0. \tag{16}$$

(With no loss of generality K_r^* may be taken 0.)
The function K^* is differentiable, monotonically increasing on $[h_r, \infty)$ and with these notations Richards' equation (1) becomes

$$\frac{\partial \theta}{\partial t} - \Delta K^*(h) + \frac{\partial k(h)}{\partial x_3} = f \text{ in } Q. \tag{17}$$

We apply C^* to the initial condition (2) and obtain

$$\theta(x, 0) = \theta_0(x) \text{ in } \Omega, \text{ where } \theta_0 := C^*(h_0)$$

and corresponding replacements should be made in the boundary conditions.
Since it is more convenient to work with the variable θ, we introduce from (14) the inverse of C^*, $(C^*)^{-1} : [\theta_r, \theta_s] \to [h_r, +\infty)$, by

$$(C^*)^{-1}(\theta) := \begin{cases} (C^*)^{-1}(\theta), \ \theta \in [\theta_r, \theta_s), \\ [0, +\infty), \ \theta = \theta_s, \end{cases} \tag{18}$$

which is multivalued at $\theta = \theta_s$, but is continuous and monotonically increasing on $[\theta_r, \theta_s)$.
Then, we replace it all over in (1) - (3).
Thus, instead of the conductivity written in function of pressure, we obtain the conductivity expressed in terms of moisture

$$K : [\theta_r, \theta_s] \to [K_r, K_s], \ K(\theta) := (k \circ (C^*)^{-1})(\theta), \ \theta \in [\theta_r, \theta_s], \tag{19}$$

function that preserves some of the properties of k, i.e., it is positive, differentiable and monotonically increasing, since for any $\theta \in [\theta_r, \theta_s)$ we have that

$$K'(\theta) = k'((C^*)^{-1}(\theta)) \cdot ((C^*)^{-1})'(\theta)) = \frac{k'((C^*)^{-1}(\theta))}{C((C^*)^{-1}(\theta))} > 0. \tag{20}$$

We notice also that

$$K'(\theta_r) = 0 \tag{21}$$

and

$$\lim_{\theta \nearrow \theta_s} K'(\theta) = +\infty, \tag{22}$$

even the limit of the derivative of k at $h = 0$, K'_0 is either infinity or a finite value. Since k is convex on $[h_r, 0)$, the same property follows for K, too,

$$K'' = \frac{k''C - kC'}{C^3} \geq 0, \text{ on } [\theta_r, \theta_s). \tag{23}$$

However, for $\theta \in [\theta_r, \theta_l]$ with $\theta_l < \theta_s$ the derivative of K is bounded, so that K follows to be Lipschitz on intervals strictly included in $[\theta_r, \theta_s)$

$$\left| K(\theta) - K(\overline{\theta}) \right| \leq M_l \left| \theta - \overline{\theta} \right|, \ \forall \theta, \overline{\theta} \in [\theta_r, \theta_l], \ \theta_l < \theta_s, \tag{24}$$

where

$$M_l = \max_{\theta \in [\theta_r, \theta_l]} \frac{k'((C^*)^{-1}(\theta))}{C((C^*)^{-1}(\theta))} < \infty. \tag{25}$$

Plugging (18) in (15) we get the function

$$\beta^*(\theta) := \begin{cases} (K^* \circ (C^*)^{-1})(\theta), \ \theta \in [\theta_r, \theta_s), \\ [K_s^*, +\infty), \ \theta = \theta_s \end{cases} \tag{26}$$

that turns out to be multivalued and notice immediately that

$$\lim_{\theta \nearrow \theta_s} \beta^*(\theta) = K_s^*. \tag{27}$$

For $\theta \in [\theta_r, \theta_s)$ the function $(C^*)^{-1}$ is monotonically increasing, so that we can calculate $\beta^*(\theta)$ by changing the variable in the integral (15), by denoting $\zeta = (C^*)^{-1}(\xi)$. In this way we get

$$\beta^*(\theta) = K_r^* + \int_{\theta_r}^{\theta} \beta(\xi) d\xi, \text{ for } \theta \in [\theta_r, \theta_s),$$

where

$$\beta(\theta) := \frac{k((C^*)^{-1}(\theta))}{C((C^*)^{-1}(\theta))}, \text{ for } \theta \in [\theta_r, \theta_s). \tag{28}$$

In this way we have rigorously recovered the definition of the water diffusivity function
We notice that β has two important properties

$$\beta(\theta) \geq \rho := \beta(\theta_r) = \frac{K_r}{C_r} > 0, \ \forall \theta \in [\theta_r, \theta_s) \tag{29}$$

and

$$\lim_{\theta \nearrow \theta s} \beta(\theta_s) = +\infty. \tag{30}$$

Moreover, by the hypotheses made upon the functions C and k it follows that β is monotonically increasing and convex, i.e.,

$$\beta' = \frac{k'C - kC'}{C^3} \geq 0, \text{ on } [\theta_r, \theta_s), \tag{31}$$

$$\beta'(\theta_r) = 0, \tag{32}$$

$$\beta'' = \frac{(k''C - kC'')C - 2C'(k'C - kC')}{C^4} > 0, \text{ on } [\theta_r, \theta_s). \tag{33}$$

Hence, β^* is three times differentiable, monotonically increasing and convex on $[\theta_r, \theta_s)$ and as a matter of fact we can write

$$\beta^*(\theta) = \begin{cases} K_r^* + \int_{\theta_r}^{\theta} \beta(\xi)d\xi \text{ for } \theta \in [\theta_r, \theta_s), \\ [K_s^*, +\infty) \text{ for } \theta = \theta_s. \end{cases} \tag{34}$$

Moreover, by (29) and (30) we deduce that the function β^* satisfies the inequality

$$(\beta^*(\theta) - \beta^*(\theta))(\theta - \overline{\theta}) \geq \rho(\theta - \overline{\theta})^2, \forall \theta, \overline{\theta} \in [\theta_r, \theta_s]. \tag{35}$$

In conclusion we can set

Model 1. Let us assume $(m_1) - (m_5)$, $C_0 = 0$ and $K_0' \in \mathbf{R}_+ \cup \{\infty\}$.
Then, the diffusive model of the *strongly nonlinear saturated-unsaturated infiltration with a strongly nonlinear hydraulic conductivity* is given by

$$\frac{\partial \theta}{\partial t} - \Delta\beta^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} = f \text{ in } Q, \tag{36}$$

$$\theta(x, 0) = \theta_0(x) \text{ in } \Omega, \tag{37}$$

$$\textit{boundary conditions in } \theta, \tag{38}$$

where β^* is the multivalued function defined by (34), β is given by (28) and K is the single-valued function (19). Moreover, β^* is strongly monotone, β satisfies (29)-(33) and K has the properties (21)-(25).

In particular, the boundary conditions in terms of θ are

$$(K(\theta)\mathbf{i}_3 - \nabla\beta^*(\theta)) \cdot \nu = u \text{ on } \Sigma_u = \Gamma_u \times (0, T), \tag{39}$$

$$(K(\theta)\mathbf{i}_3 - \nabla\beta^*(\theta)) \cdot \nu = \alpha\beta^*(\theta) + f_0 \text{ on } \Sigma_\alpha = \Gamma_\alpha \times (0, T). \tag{40}$$

As a matter of fact, β^* is multivalued and the sign equal (=) in (36) is not properly used, the appropriate symbol should be $\ni$. Also, we shall specify later the exact meaning of the solutions to (36)-(38). It must be emphasized that equation (36) is multivalued. This must not be surprising if one takes into account that, roughly speaking, (36) models a free boundary problem.
In order to pass to the functional approach to this model, we extend the diffusivity function to the left of θ_r by the constant ρ and the conductivity by the constant K_r and pass to the dimensionless form.

3 Existence and uniqueness of the solution to the diffusive form

The problem will be treated within the functional framework represented by $V = H^1(\Omega)$ with its dual $V' = (H^1(\Omega))'$. The norm on V is defined by

$$\|\psi\|_V = \left(\int_\Omega |\nabla\psi|^2\, dx + \int_{\Gamma_\alpha} \alpha(x)\, |\psi|^2\, d\sigma\right)^{1/2} \tag{41}$$

and it can be easily checked that it is equivalent with the standard Hilbertian norm on $H^1(\Omega)$.
We endow the dual V' with the scalar product

$$< \theta, \overline{\theta} >_{V'} := \theta(\psi),\ \forall\theta, \overline{\theta} \in V', \tag{42}$$

where $\psi \in V$ satisfies the boundary value problem

$$-\Delta\psi = \overline{\theta},\ \frac{\partial\psi}{\partial\nu} + \alpha\psi = 0 \text{ on } \Gamma_\alpha,\ \frac{\partial\psi}{\partial\nu} = 0 \text{ on } \Gamma_u. \tag{43}$$

Definition. Let

$$\theta_0 \in L^2(\Omega),\ \theta_0 \le \theta_s \text{ a.e. } x \in \Omega,$$
$$f \in L^2(0,T;V'),\ u \in L^2(0,T;L^2(\Gamma_u)),\ f_0 \in L^2(0,T;L^2(\Gamma_\alpha)).$$

We mean by *solution* to (36)-(37), (39)-(40) a function $\theta \in C([0,T];L^2(\Omega))$, such that

$$\frac{d\theta}{dt} \in L^2(0,T;V'), \tag{44}$$

$$\theta(x,t) \le \theta_s \text{ a.e. } (x,t) \in Q, \tag{45}$$

$$\begin{aligned} < & \frac{d\theta}{dt}(t), \psi >_{V',V} + \int_\Omega \left(\nabla\eta(t)\cdot\nabla\psi - K(\theta(t))\frac{\partial\psi}{\partial x_3}\right) dx \\ = & < f(t), \psi >_{V',V} - \int_{\Gamma_\alpha} (\alpha\eta(t) + f_0(t))\psi d\sigma - \int_{\Gamma_u} u(t)\psi d\sigma, \\ \text{a.e. } t \in & \ (0,T),\ \forall\psi \in V, \\ \text{where } \eta \in & \ L^2(0,T;V) \text{ is such that } \eta(x,t) \in \beta^*(\theta(x,t)) \text{ a.e. } (x,t) \in Q, \end{aligned} \tag{46}$$

and

$$\theta(x,0) = \theta_0 \text{ in } \Omega. \tag{47}$$

We introduce now the operatorial form of (36)-(37), (39)-(40), by setting

$$D(A) := \{\theta \in L^2(\Omega);\ \exists\eta \in V,\ \eta(x) \in \beta^*(\theta(x)),\ \text{a.e. } x \in \Omega\} \tag{48}$$

and we define the multivalued operator $A : D(A) \subset V' \to V'$ by

$$\begin{aligned} < & \ A\theta, \psi >_{V',V} := \int_\Omega \left(\nabla\eta\cdot\nabla\psi - K(\theta)\frac{\partial\psi}{\partial x_3}\right) dx + \int_{\Gamma_\alpha} \alpha\eta\psi d\sigma, \\ \forall\psi \in & \ V,\ \text{for some } \eta \in \beta^*(\theta). \end{aligned} \tag{49}$$

Moreover, we define the operator $B \in L(L^2(\Gamma_u); V')$ and the function $f_\Gamma \in L^2(0,T;V')$ by

$$Bu(\psi) := -\int_{\Gamma_u} u\psi d\sigma, \ \forall \psi \in V, \tag{50}$$

$$f_\Gamma(t)(\psi) := -\int_{\Gamma_\alpha} f_0\psi d\sigma, \ \forall \psi \in V \tag{51}$$

and with these notations we introduce the Cauchy problem

$$\frac{d\theta}{dt} + A\theta \ni f + Bu + f_\Gamma, \text{ a.e. } t \in (0,T), \tag{52}$$

$$\theta(0) = \theta_0(x) \text{ in } \Omega. \tag{53}$$

Let us define the function $j : \mathbf{R} \to (-\infty, \infty]$ by

$$j(r) = \begin{cases} \int_0^r \beta^*(\xi)d\xi, \text{ if } r \leq \theta_s \\ +\infty, \text{ if } r > \theta_s, \end{cases} \tag{54}$$

where $j(\theta_s)$ should be understood as

$$j(\theta_s) = \lim_{r \nearrow \theta_s} \int_0^r \beta^*(\xi)d\xi. \tag{55}$$

It follows that j is a proper, convex, lower semicontinuous function and

$$\partial j(\theta) = \begin{cases} \beta^*(\theta), \ \theta < \theta_s \\ [K_s^*, +\infty), \ \theta = \theta_s \\ \emptyset, \ \theta > \theta_s \end{cases} \tag{56}$$

Theorem. *Let f, u, f_0 and θ_0 satisfy*

$$f \in L^2(0,T;V'), \ u \in L^2(0,T;L^2(\Gamma_u)), \ f_0 \in L^2(0,T;L^2(\Gamma_\alpha)), \tag{57}$$

$$\theta_0 \in L^2(\Omega), \ \theta_0 \leq \theta_s, \ a.e. \ x \in \Omega. \tag{58}$$

Then, there exists a unique solution θ to the exact problem (52)-(53) *with the following properties*

$$\begin{aligned} \theta &\in C([0,T];L^2(\Omega)) \cap W^{1,2}(0,T;V') \cap L^2(0,T;V), \\ \beta^*(\theta) &\in L^2(0,T;V), \ K(\theta) \in L^2(0,T;V), \\ j(\theta) &\in L^1(Q). \end{aligned} \tag{59}$$

Moreover, the solution satisfies the estimates

$$\begin{aligned} &\int_\Omega j(\theta(x,t))dx + \int_0^t \left\| \frac{d\theta}{d\tau}(\tau) \right\|_{V'}^2 d\tau + \int_0^t \|\beta^*(\theta(\tau))\|_V^2 d\tau \\ \leq \ &\gamma_0(\alpha_m) \left(\int_\Omega j(\theta_0(x))dx + \int_0^T \|f(\tau)\|_{V'}^2 d\tau \right. \\ &\left. + \int_0^T \|u(\tau)\|_{L^2(\Gamma_u)}^2 d\tau + \int_0^T \|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2 d\tau \right), \end{aligned} \tag{60}$$

and

$$
\begin{aligned}
& \|\theta(t)\|_{V'}^2 + \int_0^t \|\theta(\tau)\|^2 \, d\tau \\
\leq \; & \gamma_1(\alpha_m) \Bigg(\|\theta_0\|_{V'}^2 + \int_0^T \|f(\tau)\|_{V'}^2 \, d\tau \\
& + \int_0^T \|u(\tau)\|_{L^2(\Gamma_u)}^2 \, d\tau + \int_0^T \|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2 \, d\tau \Bigg).
\end{aligned}
\tag{61}
$$

Here γ_0, $\gamma_1 = O(1/\alpha_m)$, as $\alpha_m \to 0$. The proof of this theorem presumes the proof of an approximate problem introduced by replacing the multivalued function $\beta^*(\theta)$ by a smooth approximation of it $\beta^*(\theta)$. details of the proof can be found in [2] and [3].

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