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Finite element-boundary element approach of MHD Pipe Flow

by

Emil LUNGU and Alin POHOATA [1]

This paper deals with the flow of a viscous conducting fluid in a pipe with arbitrary cross-section and arbitrary wall conductivities under the influence of a transverse magnetic field. For the numerical solution a finite element discretization is considered in the domain corresponding to the fluid and inside the walls of the pipe. When the outer medium is considered with an arbitrary conductivity the finite element method is coupled with the boundary element method. The proposed method is illustrated with numerical example.

Key words and phrases: MHD pipe flow, finite element method, boundary element method

1 Introduction

The flow of an incompressible, viscous, electrically conducting fluid through a cylindrical duct under the influence of an uniform transverse magnetic field may be accelerated or slowed by changing the intensity of this field. Also, the presence of the magnetic field give rise of an induction current which may be captured in exterior. Therefore, the study of the equations that models this phenomenon is of considerable practical interest having many direct application as power generation, magnetic propulsion devices, electromagnetic pumps, induction flow meters, electrolysis processes.

This paper deals with a numerical investigation of the influence of conducting wall to the fluid flow. This problem extends the particular cases of a flow through a pipe having perfectly insulating or perfectly conducting walls. For these cases the problem is reduced to a system of partial differential equations in the domain occupied by the fluid. For pipe with circular or rectangular cross-section analytical solutions are available. A general presentation of the problem may be found in [4]. Numerical approximations for the equations inside the pipe are presented in [1], [7], [2]. In [1] a numerical solution based on the boundary element method is considered. A finite element solution is given in [7]. A.J.Meir also presents theoretical results concerning the existence and uniqueness of the weak solution. Error estimations are also established. A numerical solution based on pseudospectral (PS) collocation method is proposed in [2]. The same method is used for investigating the influence of the conducting wall of the pipe over the flow. For tacking into account this influence the Navier-Stokes equations and the magnetic induction equation were considered inside the pipe and the magnetic induction equation

[1] Universitatea Valahia Târgovişte, Bd. Unirii 18-24, Târgovişte, ROMÂNIA
E-mail: `emil_lungu@hotmail.com ; alinpohoata@yahoo.com`

in the wall. Proper transmission conditions were also considered along the interface between the fluid and the wall of the pipe. The application of the PS method was restricted to the case of circular cross-section. In [3] the problem is further extended to the investigation of the magnetic induction in whole space. The PS discretisation inside the domains corresponding to the fluid and the walls is coupled with the boundary element discretisation in the exterior domain. In this article we extend the work presented in [7] by applying the finite element method to the domain corresponding to the wall of the pipe. Moreover, we propose a coupled finite element - boundary element method for approximating the solution of the real problem, that is the exterior medium is considered with a certain conductivity. The remaining part of the paper is organized as follow. Section 2 presents the problem. In section 3 we present the discretisation by finite element method for the problem of fluid flow through a pipe in an insulating medium.The numerical results are compared with the solution obtained by PS method. In section 4 we present the boundary element for discretizing the equation in the exterior medium and make the coupling with the finite element method for the domains corresponding to the pipe and the fluid.

2 The statement of the problem

We consider a straight cylindrical duct with constant thickness walls of sufficient length, so that the end effects may be neglected. We assume that the fluid flowing through is viscous, incompressible and has electrical permittivity and magnetic permeability close to those of the external space ($\varepsilon \approx \varepsilon_0$, $\mu \approx \mu_0$). The relation $\mu \approx \mu_0$ is also considered inside the wall of the duct. The non-dimensional magnetic induction and electric field intensity at infinity, are supposed to be perpendicular to the axis of the duct. We assume that the Oz -axis is coincident with the duct axis and the Ox - axis is parallel with the magnetic induction at infinity. We denote by Ω_1 the region occupied by the fluid, by Ω_2 the wall region and by Ω_3 the outside region. We also denote by $\Gamma_{ij} = \partial\Omega_i \bigcap \partial\Omega_j,\ i \neq j$ and $\Omega = \overline{\Omega_1} \bigcup \Omega_2$. The steady magnetofluid dynamics equations in non-dimensional variables are:

- The equation of continuity

$$div\mathbf{V} = 0 \tag{1}$$

- The Navier - Stokes equations of motion

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = -grad\ p + \frac{1}{R_e}\Delta\mathbf{V} + R_h\mathbf{J} \times \mathbf{B} \tag{2}$$

- Maxwell's equations

$$curl\ \mathbf{E} = 0,\ div\ \mathbf{B} = 0,\ curl\ \mathbf{E} = 0, div\ \mathbf{E} = 0 \tag{3}$$

- Ohm's law

$$\mathbf{J} = R_m(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \tag{4}$$

In the above equations we denoted by $\mathbf{B} = (B_x, B_y, B_z)$ non-dimensional magnetic induction, $\mathbf{E} = (E_x, E_y, E_z)$ non-dimensional electric field intensity, $\mathbf{J} = (J_x, J_y, J_z)$ intensity of the electric conducting current, p non-dimensional pressure, R_e Reynolds number, R_h magnetic pressure number, R_m magnetic Reynolds number and $\mathbf{V} = (V_x, V_y, V_z)$ non-dimensional fluid velocity. To these equations we add the no-slip condition $\mathbf{V}|_{\Gamma_{12}} = 0$ and jump conditions

$$[\mathbf{E}]_{\Gamma_{12}} \cdot \mathbf{s} = 0,\ [\mathbf{B}]_{\Gamma_{12}} \cdot \mathbf{s} = 0,\ [\mathbf{B}]_{\Gamma_{12}} \cdot \mathbf{n} = 0,\ [\mathbf{J}]_{\Gamma_{12}} \cdot \mathbf{n} = 0 \tag{5}$$

$$[\mathbf{E}]_{\Gamma_{23}} \cdot \mathbf{s} = 0,\ [\mathbf{B}]_{\Gamma_{23}} \cdot \mathbf{s} = 0,\ [\mathbf{B}]_{\Gamma_{23}} \cdot \mathbf{n} = 0,\ [\mathbf{J}]_{\Gamma_{23}} \cdot \mathbf{n} = 0 \tag{6}$$

where $\mathbf{n}$ and $\mathbf{n}$ are the normal and tangential vectors to the corresponding interfaces. Assuming that the fluid motion is due to a pressure gradient along the pipe we may write

$$\frac{\partial}{\partial z}(\mathbf{V}, \mathbf{B}) = 0, \quad \frac{\partial p}{\partial z} = -\frac{1}{R_e} \tag{7}$$

In addition we assume that the velocity field has only a z component, that is

$$\mathbf{V} = (0, 0, V_z) \tag{8}$$

and this is consistent with the continuity equation.
These assumptions allow us to reduce the problem to the following system of equations

$$\frac{1}{R_e}\Delta\mathbf{V} + R_h(\mathbf{B} \cdot \nabla)\mathbf{B} = P \tag{9}$$

$$\Delta\mathbf{B} + R_m(\mathbf{B} \cdot \nabla)\mathbf{V} = 0 \tag{10}$$

where $P = p + R_h\frac{\mathbf{B}^2}{2}$ is the total pressure. Equation (9) is considered in Ω_1 while (10) is valid for whole space with $\mathbf{V}$ equal to zero for the domains Ω_2 and Ω_3. In [4] is shown the way to find B_x and B_y independently of V_z and B_z. For the unknown components (V_z and B_z) the index z will be omitted and we note that they are functions of x and y only. We will also use the same notation for the projection of the domain $\Omega_i,\ i = 1, 2, 3$ onto xOy plane. We now have to solve the following system of partial differential equations

$$\Delta V(x, y) + R_e R_h \frac{\partial B_1}{\partial x} = -1 \quad (x, y) \in \Omega_1 \tag{11}$$

$$\Delta B_1(x, y) + R_{m_1}\frac{\partial V}{\partial x} = 0 \quad (x, y) \in \Omega_1 \tag{12}$$

$$\Delta B_2(x, y) = 0 \quad (x, y) \in \Omega_2 \tag{13}$$

$$\Delta B_3(x, y) = 0 \quad (x, y) \in \Omega_3 \tag{14}$$

with conditions

$$V(x, y) = 0 \quad (x, y) \in \Gamma_{12} \tag{15}$$

$$B_1(x, y) = B_2(x, y) \quad (x, y) \in \Gamma_{12} \tag{16}$$

$$\frac{1}{R_{m_1}}\frac{\partial B_1}{\partial n_1}(x,y) = -\frac{1}{R_{m_2}}\frac{\partial B_2}{\partial n_2}(x,y) \qquad (x,y)\in\Gamma_{12} \tag{17}$$
$$B_2(x,y) = B_3(x,y) \quad (x,y)\in\Gamma_{23} \tag{18}$$
$$\frac{1}{R_{m_2}}\frac{\partial B_2}{\partial n_2}(x,y) = -\frac{1}{R_{m_3}}\frac{\partial B_3}{\partial n_3}(x,y) \qquad (x,y)\in\Gamma_{23} \tag{19}$$
$$\lim_{x^2+y^2\to\infty} B_3(x,y) = 0 \tag{20}$$

where n_i is the unit outward normal vector with respect to $\partial\Omega_i$. For details about the way to obtain the system and conditions the reader may consult [4] and [2].

3 Domain Decomposition Finite Element Discretisation

Let us now consider the case of a pipe with an arbitrary conductivity in a perfectly insulating environment, which implies that $B_3 = 0$. Now, our model problem consists of equations (11), (12), (13) aside the interface conditions (16), (17) and boundary conditions (15) and $B_2(x,y) = 0$, $(x,y)\in\Gamma_{23}$. So we can split the reduced problem in two subproblems. First we have to solve the Dirichlet problem Find V such that :

$$\begin{cases} -\frac{1}{R_eR_h}\Delta V(x,y) = \frac{\partial B_1}{\partial x}(x,y) + \frac{1}{R_eR_h} & (x,y)\in\Omega_1 \\ V(x,y) = 0 & (x,y)\in\Gamma_{12} \end{cases} \tag{21}$$

and a domain decomposition problem:
Find B_1 and B_2 such that

$$\begin{cases} -\frac{1}{R_{m1}}\Delta B_1(x,y) = \frac{\partial V}{\partial x}(x,y) & (x,y)\in\Omega_1 \\ -\frac{1}{R_{m2}}\Delta B_2(x,y) = 0 & (x,y)\in\Omega_2 \\ B_1(x,y) = B_2(x,y) & (x,y)\in\Gamma_{12} \\ B_2(x,y) = 0 & (x,y)\in\Gamma_{23} \\ \frac{1}{R_{m1}}\frac{\partial B_1}{\partial n_1}(x,y) + \frac{1}{R_{m2}}\frac{\partial B_2}{\partial n_2}(x,y) = 0 & (x,y)\in\Gamma_{12} \end{cases} \tag{22}$$

Using a standard nonoverlaping domain decomposition argument problem (22) is equivalent in the weak sense to (see [9] [10])

$$\begin{cases} \beta(x,y)\Delta B(x,y) = g(V,x,y) & (x,y)\in\Omega \\ B(x,y) = 0 & (x,y)\in\partial\Omega \end{cases} \tag{23}$$

where

$$g(V,x) = \begin{cases} \frac{\partial V}{\partial x}(x,y) & \text{for } (x,y)\in\Omega_1 \\ 0 & \text{for } (x,y)\in\Omega_2 \end{cases}, \tag{24}$$

β is a piecewise constant function $\beta(x,y) = \frac{1}{R_{mi}}$ for $(x,y)\in\Omega_i$ and $B|_{\Omega_i} = B_i$ for i=1,2.
Let us now consider that we have a quasi-uniform triangulations T_h on Ω and $T_{i,h}$ on Ω_i in such a way that $T_h = T_{1,h}\cup T_{2,h}$.

Let

$$S_h^1(\Omega) = span\{\phi_k^1\}_{k=1}^{\bar{N}} \subset H_0^1(\Omega)$$

be the global finite element spaces of piecewise linear and continuous basis functions ϕ_k^1 which vanish on the domain boundary $\partial\Omega$,

$$S_h^1(\Omega_i) = span\{\phi_{k,i}^1\}_{k=1}^{\bar{M}_i} \subset H_0^1(\Omega_i)$$

for i=1,2 be the local finite element spaces of piecewise linear and continuous basis functions $\phi_{k,i}^1$ which vanish on the subdomains boundary $\partial\Omega_i$ and

$$S_h^1(\Gamma_{12}) = span\{\varphi_k^1\}_{k=1}^{M} \subset S_h^1(\Omega)$$

be a conformal finite dimensional trial space of piecewise linear continuous basis functions φ_k^1 generated by the discretisation of the internal boundary. We observe that

$$S_h^1(\Omega) = S_h^1(\Omega_1) \oplus S_h^1(\Omega_2) \oplus S_h^1(\Gamma_{12}) \tag{25}$$

The Galerkin discretisation of our problem leads to the algebraic system

$$\begin{pmatrix} \frac{1}{R_eR_h}K_{II}^1 & D_{II} & 0 & D_{I\Gamma} \\ D_{II} & \frac{1}{R_{m1}}K_{II}^1 & 0 & \frac{1}{R_{m1}}K_{I\Gamma}^1 \\ 0 & 0 & \frac{1}{R_{m2}}K_{II}^2 & \frac{1}{R_{m2}}K_{I\Gamma}^2 \\ D_{\Gamma I} & \frac{1}{R_{m1}}K_{I\Gamma}^{1}{}^\top & \frac{1}{R_{m2}}K_{I\Gamma}^{2}{}^\top & K_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \underline{V} \\ B_{I,1} \\ \underline{B}_{I,2} \\ \underline{B_\Gamma} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{26}$$

with block matrices defined by

$$K_{II}^i[m,n] := \int_{\Omega_i} \nabla\phi_{n,i}^1(x,y)\nabla\phi_{m,i}^1(x,y)dxdy \tag{27}$$

$$K_{I\Gamma}^i[m,k] := \int_{\Omega_i} \nabla\varphi_{k,i}^1(x,y)\nabla\phi_{m,i}^1(x,y)dxdy \tag{28}$$

$$K_{\Gamma\Gamma}^i[l,k] := \int_{\Omega_i} \frac{1}{R_{m_i}}\nabla\varphi_{k,i}^1(x,y)\nabla\varphi_{l,i}^1(x,y)dxdy \tag{29}$$

$$K_{\Gamma\Gamma} = K_{\Gamma\Gamma}^1 + K_{\Gamma\Gamma}^2 \tag{30}$$

for all $k,l = 1$,..., M and $m,n = 1$,..., $\bar{M}_i$, i=1,2. The matrices D_{II}, $D_{\Gamma I}$ and $D_{I\Gamma}$ are defined as follows:

$$D_{II}[n,m] = \int_{\Omega_1} \frac{\partial\phi_{m,1}^1}{\partial x}(x,y)\phi_{n,1}^1(x,y)dxdy \tag{31}$$

$$D_{\Gamma I}[k,m] := \int_{\Omega_i} \frac{\partial\phi_{m,1}^1(x,y)}{\partial x}\varphi_k^1(x,y)dxdy \tag{32}$$

$$D_{I\Gamma}[m,k] := \int_{\Omega_i} \frac{\partial\varphi_k^1(x,y)}{\partial x}\phi_{m,1}^1(x,y)dxdy \tag{33}$$

for all $k = 1, \ldots, M$ and $m, n = 1, \ldots, \bar{M}_1$. For simplicity we denoted by $\Gamma = \Gamma_{12}$. The right hand side is defined by

$$f_{(m)} := \frac{1}{R_e R_h} \int_{\Omega_1} \phi^1_{m,1}(x, y) dx dy \tag{34}$$

In the system (26) the first row corresponds to the finite element discretisation of the problem (21) and the next three rows correspond to the discretisation of the problem (23).
Example: We consider the domains $\Omega_1 = \{(x, y) \in \mathbf{R}^2 : \; x^2 + y^2 \leq 1\}$, $\Omega_2 = \{(x, y) \in \mathbf{R}^2 : \; 1 \leq x^2 + y^2 \leq 2.25\}$ and take the parameters of the problem as follow $R_e = 1$, $R_h = 10$, $R_{m_1} = 10$ and $R_{m_2} = 1$. Figures (1), (2) show the approximate V_z and B_z respectively. Table 1 shows a comparison between the F.E. and P.S. approximated solutions of B_z in a set of points along the semi-axis Ox.

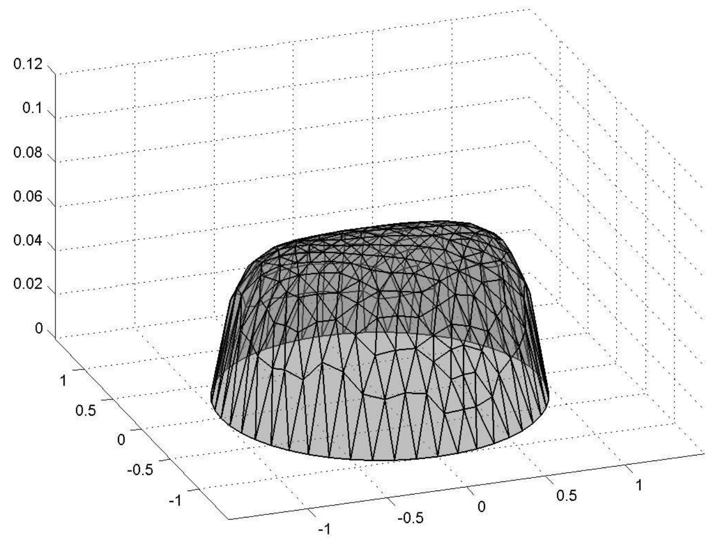

Figure 1: V_z

x	B_z (FEM)	B_z (PS)	x	B_z (FEM)	B_z (PS)
0.07473	-0.00674	-0.00673	1.07322	-0.0162	-0.01624
0.36534	-0.03269	-0.03265	1.1365	-0.01332	-0.01333
0.62349	-0.05358	-0.05344	1.21089	-0.01019	-0.0102
0.82624	-0.05958	-0.0592	1.28911	-0.00715	-0.00717
0.95557	-0.03803	-0.03888	1.3635	-0.00449	-0.0045
1	-0.0199	-0.01996	1.42678	-0.00235	-0.00236
1.00308	-0.01975	-0.0198	1.47275	-0.00087	-0.00086
1.02725	-0.0185	-0.01853	1.49692	-0.00009	-0.0001

Table 1: Approximated B_z solution along the semi-axis Ox

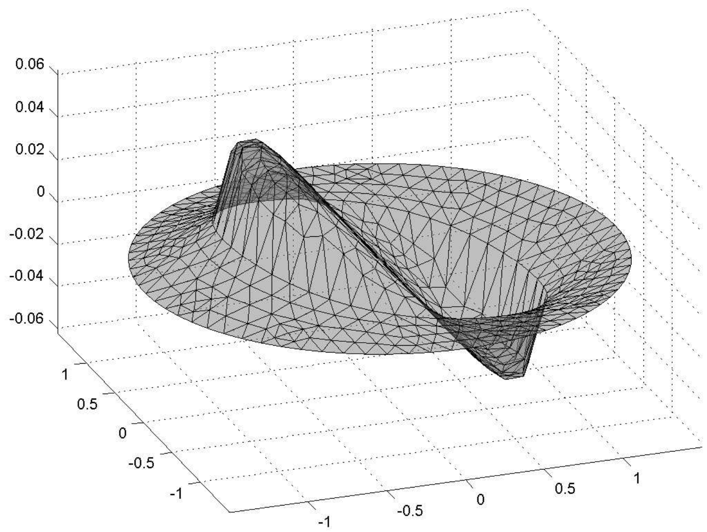

Figure 2: B_z

4 FEM-BEM Coupling

Let us now come back to the original problem (11)-(20). Since B_i satisfy Laplace equation on $\Omega_i,\ i = 2,3$ and because of (20) they can be written by using the representation formulae (see [8])

$$B_i(P) = \int_{\partial\Omega_i} U^*(P,Q)\frac{\partial}{\partial n_i}B_i(Q)ds_Q - \int_{\partial\Omega_i} \frac{\partial}{\partial n_{i(Q)}}U^*(P,Q)B_i(Q)ds_Q \tag{35}$$

for $P(x,y) \in \Omega_i$. Here $U^*(P,Q)$ is the fundamental solution of the Laplace operator:

$$U^*(P,Q) = \ -\tfrac{1}{2\pi}\log(|\overrightarrow{PQ}|) = -\tfrac{1}{2\pi}\log(\sqrt{(x-\xi)^2+(y-\eta)^2}) \quad . \tag{36}$$

with $P = P(x,y)$ and $Q = Q(\xi,\eta)$. On the boundary $\partial\Omega_i$, i=2,3 the solution verifies the Cauchy-Calderon equation

$$\begin{pmatrix} B_i \\ t_i \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K_i & V_i \\ D_i & \frac{1}{2}I + K_i' \end{pmatrix} \begin{pmatrix} B_i \\ t_i \end{pmatrix} \tag{37}$$

where $t_i = \frac{\partial}{\partial n_i}B_i$ is the normal derivative on $\partial\Omega_i$, and the boundary integral operators are given as, the single layer potential

$$(V_i t_i)(P) := \int_{\partial\Omega_i} U^*(P,Q)t_i(Q)ds_Q,\ P \in \partial\Omega_i,$$

the double layer potential

$$(K_i B_i)(P) := \int_{\partial\Omega_i} \frac{\partial}{\partial n_{i(Q)}}U^*(P,Q)B_i(Q)ds_Q,\ P \in \partial\Omega_i,$$

the adjoint double layer potential

$$(K_i' t_i)(P) := \int_{\partial\Omega_i} \frac{\partial}{\partial n_{i(P)}} U^*(P,Q) t_i(Q) ds_Q, \ P \in \partial\Omega_i,$$

and the hypersingular boundary integral operator

$$(D_i B_i)(P) := -\frac{\partial}{\partial n_{i(P)}} \int_{\partial\Omega_i} \frac{\partial}{\partial n_{i(Q)}} U^*(P,Q) B_i(Q) ds_Q, \ P \in \partial\Omega_i.$$

The properties of all boundary integral operators are wellknown (see for example [8]). In particular, the local single layer potential V_i is positive definite in the two dimensional case when we assume $diam(\Omega_i) < 1$.
From (37) we obtain the local Dirichlet-Neumann map

$$t_i(P) := [D_i + (\frac{1}{2}I + K_i')V_i^{-1}(\frac{1}{2}I + K_i)]B_i(P) =: (S_i B_i)(P) \quad \text{for} \ \ P \in \partial\Omega_i, \tag{38}$$

where $S_i : H^{1/2}(\partial\Omega_i) \longrightarrow H^{-1/2}(\partial\Omega_i)$ denotes the local Steklov-Poincaré operator.
Now we consider the transmission conditions of the functions B_i and of the conormal derivate $\alpha_i t_i$ along Γ_{23}.

$$\frac{1}{R_{m_2}}(S_2 B_2)(P) + \frac{1}{R_{m_3}}(S_3 B_3)(P) = 0 \quad \text{for } P \in \Gamma_{23} \tag{39}$$

The variational problem is to find $B \in H^{1/2}(\Gamma_{2,3})$ such that

$$\sum_{i=2}^{3} \int_{\Gamma_{2,3}} \frac{1}{R_{m_i}} (\tilde{S}_i u)(Q) v(Q) ds_Q = 0 \tag{40}$$

for all $v \in H_0^{1/2}(\Gamma_{23})$.
The Galerkin discretization of the problem (40) with boundary elements in Ω_3 and finite elements in Ω_2 yields to the linear system

$$S_{2,h}^{FEM} \underline{B}_{\Gamma_{23}} + S_{3,h}^{BEM} \underline{B}_{\Gamma_{23}} = 0, \tag{41}$$

The matrices $S_{i,h}^{FEM/BEM}$ are nothing else than the discretized version of the Steklov-Poincaré operator by FEM or BEM. For the implementation of discretized matrices of the Steklov-Poincaré operator see [5] and [6].

5 Conclusions

We proposed a numerical solution based on domain decomposition techniques. We used finite element discretisations on the internal subdomains and boundary element on the external subdomain. The coupling was realised using the discrete approximations of the local Steklov-Poincaré operators. We observe that our solution was very close to the solution computed with PS in ([2]). The proposed method is preferred if we consider a more general choice of the pipe cross-section.

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