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The Study of a Laminar Non-Stationary Gravific Flow of a Viscous Fluid Between Non-Axial Cylinders

by
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Abstract

This paper deals with the study of laminar non-stationary flow of a viscous fluid between non-axial cylinders. We are using the mediation method in Navier-Stokes equation. The problem is reduced to a stationary one for which the conform domain transformation in a circular corona can be applied. For this problem, the solution is determined by using the variables separation method. The flow is accepted for different forms of the pressure gradient $\left(\frac{\partial p}{\partial z}\right)$: linear, exponential study and stability analysis

Key words and phrases: Navier-Stokes, stationary , non-stationary, conformable mapping, averaging

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1 The non-stationary case study

We are considering the non-stationary movement of a viscous incompressible fluid between two non-axial cylinders, see figure 1. The equations of the viscous fluid's laminar movement given by Navier-Stokes, in which are considered the gravic force and the difference of a constant pressure generated by a certain pump ,$\frac{\partial p}{\partial z} = -f(t) \equiv k$, are:

$$\nu \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right] = \frac{\partial w}{\partial t} + \frac{1}{\rho}\left[\frac{\partial p}{\partial z} - \rho g \sin\alpha\right] \tag{1}$$

The initial and boundary conditions are:

$$\begin{cases} w(r,\theta,t=0) = 0 \\ w(r,\theta,t)_C = w(r,\theta,t)_\gamma = 0 \end{cases} \tag{2}$$

wher C and γ are the contours of circles.

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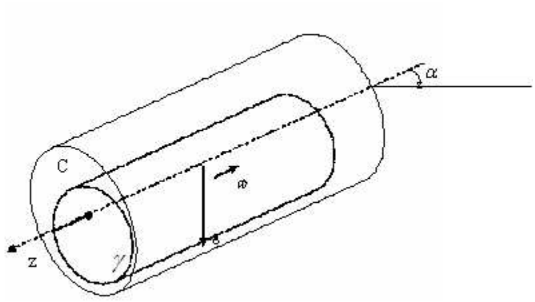

Figure 1: Two non-axial cylinders between which the viscous fluid flows with laminar speed w

The flow is ensured by the incline plane and by the pump.
We are using the averaging method Slezkin-Targ [5]:

$$W(t) = \frac{1}{A_D} \int \int_D \frac{\partial w}{\partial t} dxdy \tag{3}$$

We introduce (3) in (1) and obtain:

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} = G(t) \tag{4}$$

where $G(t) = \frac{1}{\nu} \frac{\partial w}{\partial t} + \frac{1}{\rho \nu} \left[\frac{\partial p}{\partial z} - \rho g \sin \alpha \right]$, considering $\rho \nu = \mu$, where ρ is the fluid density, μ the dynamic viscosity, and ν the kinematic viscosity.
We apply the averaging over $\frac{\partial w}{\partial t}$ term and obtain:

$$G(t) = \frac{1}{\nu} \frac{\partial W}{\partial t} + \frac{1}{\mu} \left[\frac{\partial p}{\partial z} - \rho g \sin \alpha \right] \tag{5}$$

We wish to eliminate $G(t)$ in order to obtain $\Delta w = 0$. Given the following substitution:

$$w = v + \frac{G(t)}{2} r^2 \sin^2 \theta \tag{6}$$

By replacing in (4) the partial derivates we obtain the homogeneous equation in v

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} = 0 \tag{7}$$

The boundary conditions (2) become:

$$v_\gamma = -\frac{G(t)}{2} r^2 \sin^2 \theta; v_C = -\frac{G(t)}{2} r^2 \sin^2 \theta \tag{8}$$

In the initial portrait of the two cylinders see figure 2, $C\left(O_1, r_1\right), C\left(O_2, r_2\right),\ r_1 < r_2$ we say that $OO_1 = d$.

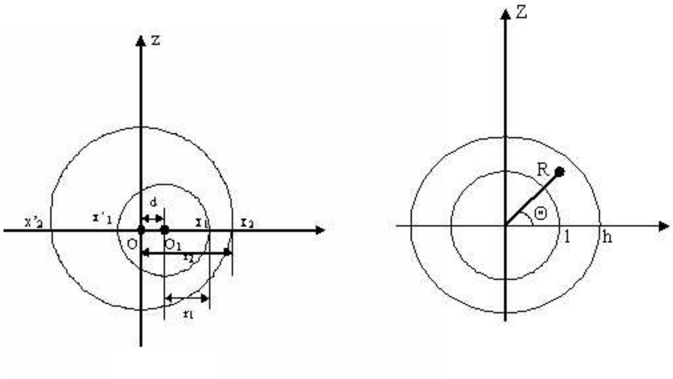

Figure 2: The portrait of the two non-axial cylinders

Due to the fact that the cylinders are non-axial we have to aply an homographic conformable mapping in order to obtain concentric cylinders [2]:

$$Z = \frac{Mz + N}{Pz + Q} = Re^{i\Theta} \tag{9}$$

After applying the conformable mapping the cylinders become axial, so that $C\left(O_1, r_1\right) \to C\left(O, 1\right)$ şi $C\left(O_2, r_2\right) \to C\left(O, h\right)$. In order to ease the calculus we are going to make the following notations:

$$Z = \frac{(A+1)z - (x_1^{'}A + x_1)}{(A-1)z - (x_1^{'}A - x_1)}, A = \sqrt{\frac{r_2^2 - (d+r_1)^2}{r_2^2 - (d-r_1)^2}},$$

$$h = \frac{1+\sqrt{\Delta}}{1-\sqrt{\Delta}}, \Delta = \frac{(r_2 - r_1)^2 - d^2}{(r_2 + r_1)^2 - d^2}$$

We switch to polar coordinates in order to obtain the $r^2 \sin^2\theta$ product. We get the following result for y^2:

$$y_{(1,2)}^2 = \frac{R^2 \sin^2\Theta\left[2A(r_1+d) - 2A(r_1-d)\right]^2}{\left[(A-1)^2R^2 - 2(A^2-1)R\cos\Theta + (A+1)^2\right]^2} = F_{(1,2)}(\Theta)$$

which becomes:

$$y_{(1,2)}^2 = \begin{cases} F_1(\Theta), R = 1 \\ F_2(\Theta), R = h \end{cases} \tag{10}$$

Therefore:

$$F_1(\Theta) = \frac{16\sin^2\Theta d^2A^2}{\left[(A-1)^2 - 2(A^2-1)\cos\Theta + (A+1)^2\right]^2}$$

$$F_2(\Theta) = \frac{16h^2 \sin^2 \Theta d^2 A^2}{(A-1)^2 h^2 - 2(A^2-1)h\cos\theta + (A+1)^2]^2}$$

Trough the conformable mapping the equation (7) becomes:

$$\frac{\partial^2 v}{\partial R^2} + \frac{1}{R}\frac{\partial v}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} = 0 \tag{11}$$

wich allows as a particular solution

$$v_o = a \ln R + b \tag{12}$$

We use the variable separation method and search for a v of the following form: $v = X(R)Y(\Theta)$. By replacing v in (11) we get:

$$R^2 \frac{X''}{X} + R\frac{X'}{X} + \frac{Y''}{Y} = 0 \Leftrightarrow R^2 \frac{X''}{X} + R\frac{X'}{X} = -\frac{Y''}{Y} = -\lambda^2$$

We obtain the equation: $Y'' + \lambda^2 Y = 0$ having $Y = C_1 \cos(\lambda\Theta)$ as solution due to the parity $v(\Theta) = v(-\Theta)$. For the Euler equation $R^2 X'' + RX' - \lambda^2 X = 0$ with the solution: $\tilde{X} = R^n$ we find $\lambda = \pm n$ This way is obtain the general solution for (11)

$$v = -\frac{G}{2}\left[a \ln R + b + \sum_{n=1}^{\infty}\left[a_n R^n + b_n R^{-n}\right]\cos n\Theta\right] \tag{13}$$

With the help of the conditions (8) in order to determine the Fourier coefficients that are part of the solution (13), we get:

$$\begin{cases} F_1(\Theta) = b + \sum_{n=1}^{\infty} (a_n + b_n) \cos n\Theta \\ F_2(\Theta) = a \ln h + b + \sum_{n=1}^{\infty} (a_n h^n + b_n h^{-n}) \cos n\Theta \end{cases} \tag{14}$$

implying the following system:

$$b = \frac{2}{\pi} \int_0^{\pi} F_1(\Theta) d\Theta, a_n + b_n = \frac{2}{\pi} \int_0^{\pi} F_1(\Theta) \cos n\Theta d\Theta,$$

$$a_n h^n + b_n h^{-n} = \frac{2}{\pi} \int_0^{\pi} F_2(\Theta) \cos n\Theta d\Theta, a \ln h + b = \frac{2}{\pi} \int_0^{\pi} F_2(\Theta) d\Theta$$

with the help of which we find the coefficients a, b, a_n, b_n. Going back to $w = v + \frac{G}{2} r^2 \sin^2 \theta$, the moving speed of the viscous fluid between the two cylinders will be:

$$w = -\frac{G}{2} \left[a \ln R + b + \sum_{n=1}^{\infty} (a_n R^n + b_n R^{-n}) \cos n\Theta + r^2 \sin^2 \theta \right] \tag{15}$$

In order to determine the solution for (15) we are using the averaging:

$$W(t) = \frac{1}{A_D} \int\int_D \frac{\partial w}{\partial t} dxdy = \frac{1}{A_D} \int\int_D \frac{\partial v_0}{\partial t} dxdy + \frac{r^2 \sin^2 \theta}{2A_D} \int\int_D \frac{\partial G}{\partial t} dxdy \tag{16}$$

To simplify we introduce the following notation:

$$E = -\frac{1}{2} \int\int_D \left(v_0 - r^2 \sin^2 \theta \right) JdXdY \tag{17}$$

Therefore the equation (16) becomes

$$W = -\frac{W'}{A_D} E \tag{18}$$

with the solution given by $W = Ce^{-\frac{A_D}{E}} t$. We place the initial conditions and get $W(0) = C$. In order to determine the constant we go back to (5) in which $G(0) = 0$. In this context we obtain $C = \nu\mu \left[f(0) - \rho g \sin \alpha \right]$. The solution for equation (18) is therefore

$$W(t) = \nu\mu \left[f(0) - \rho g \sin \alpha \right] e^{-\frac{A_D}{E} t} \tag{19}$$

and the term $G(t)$ will have the following form:

$$G(t) = \mu \left[f(0) - \rho g \sin \alpha \right] e^{-\frac{A_D}{E} t} + \frac{1}{\mu} \left[-f(t) - \rho g \sin \alpha \right] \tag{20}$$

In these circumstances the solution for equation (15) can be determined directly and represents the solution for the non-stationary case problem. . It can be observed that if $t \to \infty$, $G(t) \equiv \frac{1}{\mu} \left[-f(t) - \rho g \sin \alpha \right]$ the solution is stabilizing.

2 The stationary case study

We rely on same demonstrations as for the non-stationary case and we'll consider the equation (1) but in which the time dependent term is missing. So, the equation that is designated to be solved is:

$$\nu\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right]=\frac{1}{\rho}\left[\frac{\partial p}{\partial z}-\rho g\sin\alpha\right] \tag{21}$$

wich is equivalent to the equation $\Delta w=\frac{K}{\mu}$ using the substitution $K=\frac{\partial p}{\partial z}-g\sin\alpha$. Therefore, the particular solution of (21) will be:

$$w_p=\frac{K}{2\mu}r^2 \tag{22}$$

We perform the function substitution $w-w_p=W$ from which we get $\Delta W=0$. By placing the boundary conditions:

$$\begin{cases} w|_C=0\Rightarrow W|_C=-w_p|_{R=h}=-\frac{K}{2\mu}h^2 \\ w|_\gamma=0\Rightarrow W|_\gamma=-w_p|_{R=1}=-\frac{K}{2\mu} \end{cases} \tag{23}$$

the equation (21) in the new unknown function becomes:

$$\frac{\partial^2 W}{\partial r^2}+\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}=0 \tag{24}$$

Looking for a solution of the following type $W=X(r)Y(\theta)$ we get: $Y=C\cos\lambda\theta$, $X=r^n$, from which derives that $\lambda=\pm n$. Therefore, the equation's solution will be:

$$W=\sum_{n=1}^{\infty}\left(a_n r^n+b_n r^{-n}\right)\cos n\theta \tag{25}$$

We set the boundary conditions (23) in order to determine the coefficients that are part of W. Therefore:

$$\begin{cases} a_n h^n+b_n h^{-n}=\frac{2}{\pi}\int_0^\pi -\frac{K}{2\mu}h^2\cos n\theta d\theta \\ a_n+b_n=\frac{2}{\pi}\int_0^\pi -\frac{K}{2\mu}\cos n\theta d\theta \end{cases} \tag{26}$$

We get the solution of the problem for the stationary case:

$$w=\frac{K}{2\mu}r^2+\sum_{n=1}^{\infty}\left(a_n r^n+b_n r^{-n}\right)\cos n\theta \tag{27}$$

2.1 Conclusions

1. $k=0$, the flow will be gravic with the factor $-g\sin\alpha$ in the solution (24)

2. $\alpha=0$, in this situation only the pump acts over the installation and we have $K=\frac{\partial p}{\partial z}$, only the k factor is present in the solution

3. $K = \alpha = 0$, this case is not possible because the solution will be null.

These conclusions are the cases that stabilize the non-stationary solution (15) when $t \to \infty$. By following the solution determination effective numerical calculus can be made also to determine the debit $Q = \int_S \rho \cdot \vec{v} \cdot \vec{n} dA$. The mass and heat problem can be treated in the future.

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